Brief Introduction to Tensor Algebra

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I. Basic concepts

1. different coordinate systems

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2. tensor algorithm

   a. contravariant and covariant components

For basis $g_1, g_2, g_3$: 
non-coplanar condition: \((g_1 \times g_2) \cdot g_3 \neq 0\)

\[
\mathbf{u} = u^1 g_1 + u^2 g_2 + u^3 g_3 = u^i g_i \tag{1}
\]

For reciprocal basis \(g^1, g^2, g^3\):

\[
g^i \cdot g_j = \delta^i_j \tag{2}
\]

\[
\mathbf{u} = u_1 g^1 + u_2 g^2 + u_3 g^3 = u^i g_i \tag{3}
\]

\[
\mathbf{u} \cdot g^j = u^i g_i \cdot g^j = u^i \delta^j_i = u^j \tag{4}
\]

\[
\mathbf{u} \cdot g_j = u^i g^i \cdot g_j = u_i \delta^i_j = u_j \tag{5}
\]

\(u^i\): contravariant components

\(u_i\): covariant components

\(g^i\): contravariant base vectors

\(g_i\): covariant base vectors

b. Relationship between contravariant and covariant components

\[
u^i = \mathbf{u} \cdot g^i = (u_k g^k) \cdot g^i = u_k (g^k \cdot g^i) \tag{6}
\]

\[
u_i = \mathbf{u} \cdot g_i = (u_k g^k) \cdot g_i = u^k (g_k \cdot g_i) \tag{7}
\]

Introducing metric tensor:

\[
g_{ik} = g_i \cdot g_k, \tag{8}
\]

\[
g^{ik} = g^i \cdot g^k \tag{9}
\]

The relationship between contravariant and covariant components:

\[
u^i = g^{ik} u_k \tag{10}
\]

\[
u_i = g_{ik} u^k \tag{11}
\]
The relationship between contravariant and covariant base vectors:

\[ g_i = g_{ij} g^j \]  
\[ g^i = g^{ij} g_j \]  
\[ g_{ik} = g_i g_k = \frac{\partial y_j}{\partial x^i} \frac{\partial y_j}{\partial x^k} \]  
\[ g_{ij} = (g_{rs} g_{mn} - g_{rn} g_{ms}) / g \]  
\[ i = 1 : r = 2, m = 3; \quad j = 1 : s = 2, n = 3 \]  
\[ i = 2 : r = 3, m = 1; \quad j = 2 : s = 3, n = 1 \]  
\[ i = 3 : r = 1, m = 2; \quad j = 3 : s = 1, n = 2 \]
where \( g \) is the determinant of the metric tensor.

The Jacobian value of the coordinate transformation:

\[
J = \begin{vmatrix}
\frac{\partial y_1}{\partial x^1} & \frac{\partial y_1}{\partial x^2} & \frac{\partial y_1}{\partial x^3} \\
\frac{\partial y_2}{\partial x^1} & \frac{\partial y_2}{\partial x^2} & \frac{\partial y_2}{\partial x^3} \\
\frac{\partial y_3}{\partial x^1} & \frac{\partial y_3}{\partial x^2} & \frac{\partial y_3}{\partial x^3}
\end{vmatrix}
= \sqrt{\det g}
\tag{25}
\]

another expression of \( g^{ik} \)

\[
g^{ik} = g^i \cdot g^k = \frac{\partial x^i}{\partial y_k} \frac{\partial x^j}{\partial y_k}
\tag{26}
\]

**Some examples:**

Rectangular cartesian coordinates,

\( g_{ij} = \delta_{ij} \).

Cylindrical polar coordinates,

\( g_{11} = 1, g_{22} = \rho^2, g_{33} = 1 \), and \( g_{12} = g_{23} = g_{31} = 0 \).

Orthogonal curvilinear coordinates,

\( g_{ij} = 0, i \neq j \).

d. **Dot products in general coordinates**

1) Scalar products of two vectors

In generalized coordinates

\[
u \cdot v = (u^i g_i) \cdot (v^j g^j)
\tag{27}
\]

according to

\[
g_i \cdot g^j = \delta^j_i
\tag{28}
\]

we obtain:

\[
u \cdot v = u^i v_i
\tag{29}
\]

2) dot products of second order tensors
\[ T = T_{ij} g^i g^j \quad S = S^{kl} g_k g_l \]  

(30)

The dot products:

\[ T \cdot S = (T_{ij} g^i g^j) \cdot (S^{kl} g_k g_l) \]  

(31)

\[ = T_{ij} g^i g^j S^{kl} g_k g_l \]  

(32)

\[ = T_{ij} S^{ij} g^i g_l \]  

(33)

3) double dot products of second order tensors

The definition of double dot product:

\[ (a b) : (c d) = (a \cdot c) (b \cdot d) \]  

(34)

\[ T : S = (T_{ij} g^i g^j) : (S^{kl} g_k g_l) \]  

(35)

\[ = T_{ij} S^{kl} (g^i \cdot g_k) (g^j \cdot g_l) \]  

(36)

\[ = T_{ij} S^{ij} \]  

(37)

II. Differentiation of tensors

1. The differentiation of base vectors

1) derivatives of base vectors

\[ \frac{\partial g_i}{\partial x^j} = \Gamma^1_{ij} g_1 + \Gamma^2_{ij} g_2 + \Gamma^3_{ij} g_3 = \Gamma^k_{ij} g_k \]  

(38)

- \( \Gamma^k_{ij} \) Christoffel symbols of the second kind.

\[ \Gamma^k_{ij} g_k \cdot g^l = \Gamma^k_{ij} \delta^l_k = \frac{\partial g_i}{\partial x^j} \cdot g^l \]  

(39)

\[ \Gamma^k_{ij} = \frac{\partial g_i}{\partial x^j} \cdot g^k \]  

(40)
Christoffel symbols of the first kind:

\[
\frac{\partial g_i}{\partial x^j} = \Gamma_{ij}^1 g^1 + \Gamma_{ij}^2 g^2 + \Gamma_{ij}^3 g^3 = \Gamma_{ijk} g^k
\]

(41)

Formulas for the Christoffel symbols of the first kind are

\[
\Gamma_{ijk} = \frac{\partial g_i}{\partial x^j} \cdot g_k
\]

(42)

**Calculation of Christoffel symbols**

\[
\frac{\partial g_i}{\partial x^j} = \frac{\partial}{\partial x^j} \frac{\partial r}{\partial x^i} = \frac{\partial^2 r}{\partial x^j \partial x^i}
\]

(43)

and

\[
g^k = \frac{\partial x^k}{\partial y_l}
\]

(44)

\[
\Gamma_{ij}^k = \frac{\partial^2 y_l}{\partial x^i \partial x^j} \frac{\partial x^k}{\partial y_l}
\]

(45)

and

\[
\Gamma_{ijk} = \frac{\partial^2 y_l}{\partial x^i \partial x^j} \frac{\partial y_l}{\partial x^k}
\]

(46)

2) The relationship between the Christoffel symbols

\[
\Gamma_{ij}^k = \frac{\partial g_i}{\partial x^j} \cdot g^k = \frac{\partial g_i}{\partial x^j} \cdot g^{kl} = g^{kl} \Gamma_{ijl}
\]

(47)

similarly,

\[
\Gamma_{ijk} = g_{kl} \Gamma_{ij}^l
\]

(48)

\[
\frac{\partial g_i}{\partial x^k}?
\]

\[
g_i \cdot g^j = \delta_i^j
\]

(49)

\[
\frac{\partial g_i}{\partial x^k} \cdot g^j + g_i \cdot \frac{\partial g^j}{\partial x^k} = 0
\]

(50)

\[
\Gamma_{ik}^j = -g_i \cdot \frac{\partial g^j}{\partial x^k}
\]

(51)
\[ \frac{\partial g^j}{\partial x^k} = -\Gamma^j_{ik} g^i \] (52)

3) Examples of Christoffel symbols:

(1) Cartesian coordinates (rectangular or generalized cartesian)

\[ \Gamma^k_{ij} = \Gamma_{ijk} = 0 \] (53)

(2) Cylindrical polars

In cylindrical polar, all the components of \( \Gamma^k_{ij} \) are zero except for

\[ \Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{\rho}, \quad \Gamma^1_{22} = -\rho \] (54)

or the Christoffel symbol of the first kind,

\[ \Gamma_{122} = \Gamma_{212} = \rho, \quad \Gamma_{221} = -\rho \] (55)

(3) Spherical polars

non-zero components of Christoffel symbols for spherical polar coordinates are

\[ \Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r}, \quad \Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r} \] (56)

\[ \Gamma^1_{22} = -r, \quad \Gamma^3_{23} = \Gamma^3_{32} = \cos \theta, \quad \Gamma^1_{33} = -r \sin^2 \theta, \quad \Gamma^2_{33} = -r \sin \theta \cos \theta \] (57)

2. Covariant derivatives of vector fields

\[ \frac{\partial u}{\partial x^j} = u^i \frac{\partial g_i}{\partial x^j} + \frac{\partial u^i}{\partial x^j} g_i \]

\[ = u^i \Gamma^k_{ij} g_k + \frac{\partial u^i}{\partial x^j} g_i = (\frac{\partial u^i}{\partial x^j} + \Gamma^i_{kj} u^k) g_i \] (59)

The covariant derivative of the contravariant vector \( u^i \).

\[ u^i_{,j} = \frac{\partial u^i}{\partial x^j} + \Gamma^i_{kj} u^k \] (60)

\[ \frac{\partial u}{\partial x^j} = u^i \frac{\partial g^i}{\partial x^j} + \frac{\partial u_i}{\partial x^j} g^i \]

\[ = -u_i \Gamma^i_{kj} g^k + \frac{\partial u_i}{\partial x^j} g^i = (\frac{\partial u_i}{\partial x^j} - \Gamma^i_{jk} u^k) g^i \] (62)
The covariant derivative of the covariant vector $u_i$

$$ u_{i,j} = \frac{\partial u_i}{\partial x^j} - \Gamma^k_{ij} u_k \quad (63) $$

The relationship between these two kinds of covariant derivatives:

$$ \frac{\partial \mathbf{u}}{\partial x^j} = u_{i,j} g_i^j = u_{i}^{j} g_i = u_{j}^{i} g_i g^k = u_{j}^{k} g_k g^i \quad (64) $$

So

$$ u_{i,j} = g_{ik} u_{j}^{k} \quad (65) $$

3. Grad, div and curl

1) gradient of a scalar

The grad operator

$$ \text{grad} = \nabla = g^i \frac{\partial}{\partial x^i} \quad (66) $$

The gradient of a scalar:

$$ \text{grad} \phi = \nabla \phi = g^i \frac{\partial \phi}{\partial x^i} \quad (67) $$

$$ \text{grad} \phi = g^i \frac{\partial \phi}{\partial x^i} = g^{ij} g_j \frac{\partial \phi}{\partial x^i} \quad (68) $$

2) gradient of a vector

$$ \text{grad} \mathbf{u} = \frac{\partial \mathbf{u}}{\partial x^i} g_i^j \quad (69) $$

$$ \frac{\partial \mathbf{u}}{\partial x^i} = (\frac{\partial u^k}{\partial x^i} + u^j \Gamma^k_{ij}) g_k = u^k g_k \quad (70) $$

$$ \text{grad} \mathbf{u} = u^k g_k g^i \quad (71) $$
\[ \frac{\partial \mathbf{u}}{\partial x^i} = u_k \mathbf{g}^k \quad (72) \]

\[ \operatorname{grad} \mathbf{u} = \mathbf{v}_k \mathbf{g}^k \mathbf{g}^i \quad (73) \]

3) divergence of a vector

\[ \operatorname{div} \mathbf{u} = \frac{\partial \mathbf{u}}{\partial x^i} \cdot \mathbf{g}^i \quad (74) \]

from

\[ \frac{\partial \mathbf{u}}{\partial x^i} = \left( \frac{\partial u^k}{\partial x^i} + u^j \Gamma^k_{ij} \right) \mathbf{g}_k = u^k_{,i} \mathbf{g}_k \quad (75) \]

\[ \operatorname{div} \mathbf{u} = \frac{\partial \mathbf{u}}{\partial x^i} \cdot \mathbf{g}^i = u^k_{,i} \mathbf{g}_k \cdot \mathbf{g}^i = u^i_{,i} \quad (77) \]

\[ u^i_{,i} = \frac{\partial u^i}{\partial x^i} + \omega^i_{\ell} \Gamma^\ell_{ij} \quad (79) \]

\[ \Gamma^i_{ij} = \left( \frac{\partial \ln(\sqrt{g})}{\partial x^j} \right) \quad (80) \]

so

\[ u^i_{,i} = \frac{\partial u^i}{\partial x^i} + \omega^i_{\ell} \left( \frac{\partial \ln(\sqrt{g})}{\partial x^j} \right) \quad (81) \]

\[ = \left\{ \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^j} \left( \sqrt{g} u^i \right) \right\} \quad (82) \]

\[ \operatorname{div} \mathbf{u} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \left( \sqrt{g} u^i \right) \quad (83) \]

4) divergence of a second-order tensor

A second order tensor can be written as

\[ \mathbf{T} = T^{ij} \mathbf{g}_i \mathbf{g}_j \quad (84) \]
\[ \frac{\partial T}{\partial x^k} = \frac{\partial T^{ij}}{\partial x^k} g_{ji} + T^{ij} g_{i} \Gamma_{jk}^m g_{m} \]  
\[ = \frac{\partial T^{ik}}{\partial x^k} g_{ik} + T^{ik} \Gamma_{ik}^m g_{m} g_{j} + T^{ij} g_{i} \Gamma_{jk}^m g_{m} \]  
\[ (85) \]
\[ \frac{\partial T}{\partial x^k} \cdot g^k = \frac{\partial T^{ik}}{\partial x^k} g_{ik} + T^{ik} \Gamma_{ik}^m g_{m} + T^{ij} g_{i} \Gamma_{jk}^k g_{i} \]  
\[ = \frac{\partial T^{ik}}{\partial x^k} g_{ik} + T^{jk} \Gamma_{jk}^i g_{i} + T^{ij} \Gamma_{jk}^k g_{i} \]  
\[ (86) \]
\[ \text{div} T = T_{ik} \ g_{ik} \]  
\[ (87) \]
\[ T = T_{ij} \ g_{ij} \]  
\[ (88) \]
\[ \text{div} T = g^{ik} T_{ij,k} \ g^i \]  
\[ (89) \]
\[ T = T_{ij} \ g_{ij} \]  
\[ (90) \]
\[ \text{div} T = g^{ik} T_{j,k} \ g_i \]  
\[ (91) \]
\[ \text{where} \]
\[ T_{ik}^{ij} = \frac{\partial T^{ij}}{\partial x^k} + \Gamma_{ik}^m T^{mj} + \Gamma_{im}^j T^{ik} \]  
\[ (92) \]
\[ T_{ij,k} = \frac{\partial T_{ij}}{\partial x^k} - \Gamma_{ik}^m T_{mj} - \Gamma_{jk}^m T_{im} \]  
\[ (93) \]
\[ T_{j,k}^i = \frac{\partial T_{ij}}{\partial x^k} + \Gamma_{mk}^i - \Gamma_{jk}^m \]  
\[ (94) \]
\[ 5) \text{curl of vectors} \]
\[ \text{curl} \ u = g^i \times \frac{\partial u}{\partial x^i} \]  
\[ (95) \]
\[ \text{curl} \ u = \frac{1}{\sqrt{g}} e^{ijk} u_{k,j} g_i \]  
\[ (96) \]
\[ \text{where} \ e^{ijk} \text{ are the permutation symbols.} \]
\[ e^{ijk} = \begin{cases} 
0 & \text{if any of } i, j, k \text{ are the same} \\
+1 & \text{if } (i, j, k) \text{ is a cyclic permutation of } (1, 2, 3) \\
-1 & \text{otherwise} 
\end{cases} \]  
\[ (97) \]
III. Tensor expression of basic terms in fluid mechanics

1. convective terms

The acceleration term:

\[ a = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \text{grad}) \mathbf{u} \]  
100

\[ = \frac{\partial \mathbf{u}}{\partial t} + (\text{grad} \mathbf{u}) \cdot \mathbf{u} \]  
101

\[ = \frac{\partial \mathbf{u}}{\partial t} + \text{div}(\mathbf{uu}) - \mathbf{u}(\text{div} \mathbf{u}) \]  
102

\[ = \frac{\partial \mathbf{u}}{\partial t} + \text{grad}(\frac{1}{2} \mathbf{u} \cdot \mathbf{u}) + \omega \times \mathbf{u} \]  
103

1) \((\mathbf{u} \cdot \text{grad}) \mathbf{u}\)

\[ \mathbf{u} = u^i \mathbf{g}_i \]  
104

\[ \text{grad} = \mathbf{g}^i \frac{\partial}{\partial x^i} \]  
105

\[ (\mathbf{u} \cdot \text{grad}) \mathbf{u} = u^i \frac{\partial \mathbf{u}}{\partial x^i} \]  
106

\[ = u^i u^j \mathbf{g}_j \]  
107

2) \(\text{div}(\mathbf{uu})\)

\[ \mathbf{uu} = u^i u^j \mathbf{g}_i \mathbf{g}_j \]  
108

\[ \text{div}(\mathbf{uu}) = (u^i u^k) \cdot \mathbf{g}_i \]  
109

3) \(\text{grad}(\frac{1}{2} \mathbf{u} \cdot \mathbf{u})\)

\[ \text{grad}(\frac{1}{2} \mathbf{u} \cdot \mathbf{u}) = \frac{1}{2} u^i u_i = \frac{1}{2} g_{ij} u^i u^j \]  
110

2. pressure gradient term
\[ \nabla \eta = \frac{\partial \eta}{\partial x^i} g^i \]  
\[ = \frac{\partial \eta}{\partial x^i} g^{ij} g_j = \eta^{ij} g_j \]  
\( (111) \)
\( (112) \)

where \( \eta^{ij} \) is the contravariant component of the gradient.

3. diffusion terms

\[ \text{div} T = \text{div}(\nu D) \]  
\( (113) \)

\( D \) is the rate-of-strain tensor:

\[ D = \frac{1}{2} \left[ \text{grad} u + (\text{grad} u)^T \right] \]  
\( (114) \)

\( (\text{grad} u)^T \) is the transpose of \( \text{grad} u \).

\[ \text{div} T = \frac{1}{\sqrt{g_0}} \frac{\partial \sqrt{g_0} \nu D^{ik}}{\partial x^k} a_i + \nu D^{ik} \Gamma_{jk}^i a_i \]  
\( (115) \)

Smagorinsky’s subgrid turbulence

In rectangular cartesian coordinates:

\[ \nu_s = c \Delta x \Delta y \frac{1}{2} \left| \text{grad} u + (\text{grad} u)^T \right| \]  
\( (116) \)

In curvilinear coordinates:

\[ \nu_s = c \sqrt{g_0} \Delta \xi_1 \Delta \xi_2 |D| \]  
\[ = c \sqrt{g_0} \Delta \xi_1 \Delta \xi_2 (D : D)^{1/2} \]  
\[ = c \sqrt{g_0} \Delta \xi_1 \Delta \xi_2 (D^{ij} D_{ij})^{1/2} \]  
\( (117) \)

IV. Two examples of coordinate transformations

1. Boussinesq equations in curvilinear coordinates
Boussinesq equations in rectangular cartesian coordinates:

\[ \eta_t + \nabla \cdot \mathbf{M} = 0, \]  
(118)

\[ \begin{aligned} \mathbf{M} &= (h + \eta)\mathbf{u} + (h + \eta)\left( \frac{z_a^2}{2} - \frac{1}{6}(h^2 - h\eta + \eta^2) \right) \nabla(\nabla \cdot \mathbf{u}) \\
&\quad + (h + \eta)[z_a + \frac{1}{2}(h - \eta)] \nabla[\nabla \cdot (h\mathbf{u})], \end{aligned} \]  
(119)

\[ \begin{aligned} \mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + g\nabla \eta + \mathbf{V}_1 + \mathbf{V}_2 &= 0, \end{aligned} \]  
(120)

\[ \begin{aligned} \mathbf{V}_1 &= \frac{z_a^2}{2} \nabla(\nabla \cdot \mathbf{u}_t) + z_a \nabla[\nabla \cdot (h\mathbf{u}_t)] - \nabla[\frac{1}{2}\eta^2 \nabla \cdot \mathbf{u}_t + \eta \nabla \cdot (h\mathbf{u}_t)], \end{aligned} \]  
(121)

\[ \begin{aligned} \mathbf{V}_2 &= \nabla \{ (z_a - \eta)(\mathbf{u} \cdot \nabla)[\nabla \cdot (h\mathbf{u})] + \frac{1}{2}(z_a^2 - \eta^2)(\mathbf{u} \cdot \nabla)(\nabla \cdot \mathbf{u}) \} \\
&\quad + \frac{1}{2} \nabla \{ [\nabla \cdot (h\mathbf{u}) + \eta \nabla \cdot \mathbf{u}]^2 \}. \end{aligned} \]  
(122)

1) transformation of mass equation

\[ \eta_t + \nabla \cdot \mathbf{M} = 0, \]  
(123)

where \( \mathbf{M} \) is the depth-integrated volume flux given by

\[ \begin{aligned} \mathbf{M} &= (h + \eta)\mathbf{u} + (h + \eta)\left( \frac{z_a^2}{2} - \frac{1}{6}(h^2 - h\eta + \eta^2) \right) \nabla(\nabla \cdot \mathbf{u}) \\
&\quad + (h + \eta)[z_a + \frac{1}{2}(h - \eta)] \nabla[\nabla \cdot (h\mathbf{u})], \end{aligned} \]  
(124)

Tensor invariant forms:

\[ \eta_t + \frac{1}{\sqrt{g_0}} \frac{\partial}{\partial x^k}(\sqrt{g_0}M^k) = 0, \]  
(125)
\[ M^k = (h + \eta)u^k + (h + \eta)[\frac{z_2}{2} - \frac{1}{6}(h^2 - h\eta + \eta^2)][\frac{1}{\sqrt{g_0}} \frac{\partial}{\partial x^l}(\sqrt{g_0}u^l)]^k \]
\[ + (h + \eta)[z_\alpha + \frac{1}{2}(h - \eta)][\frac{1}{\sqrt{g_0}} \frac{\partial}{\partial x^l}(\sqrt{g_0}h^l)]^k. \] (126)

2) transformation of momentum equations

\[ \mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + g \nabla \eta + \mathbf{V}_1 + \mathbf{V}_2 = 0, \] (127)

The tensor invariant form:

\[ \frac{\partial u^k}{\partial t} + g\eta^k + u^l u^k + V_1^k + V_2^k = 0, \] (128)

\[ \mathbf{V}_1 = \frac{z_2}{2} \nabla(\nabla \cdot \mathbf{u}_t) + z_\alpha \nabla[\nabla \cdot (h\mathbf{u}_t)] - \nabla[(\frac{1}{2}\eta^2 \nabla \cdot \mathbf{u}_t + \eta \nabla \cdot (h\mathbf{u}_t)], \] (129)

Tensor invariant form:

\[ V_1^k = \frac{z_2^2}{2} \left[ \frac{1}{\sqrt{g_0}} \frac{\partial}{\partial x^l}(\sqrt{g_0}u^l) \right]^k \]
\[ + z_\alpha \left[ \frac{1}{\sqrt{g_0}} \frac{\partial}{\partial x^l}(\sqrt{g_0}h^l) \right]^k \]
\[ - \left[ \frac{\eta^2}{2\sqrt{g_0}} \frac{\partial}{\partial x^l}(\sqrt{g_0}u^l) \right] + \eta \left[ \frac{\partial}{\partial x^l}(\sqrt{g_0}h^l) \right]^k, \] (130)

\[ \mathbf{V}_2 = \nabla \{(z_\alpha - \eta)(\mathbf{u} \cdot \nabla)[\nabla \cdot (h\mathbf{u})] + \frac{1}{2}(z_2^2 - \eta^2)(\mathbf{u} \cdot \nabla)(\nabla \cdot \mathbf{u}) \}
\[ + \frac{1}{2} \nabla \{[\nabla \cdot (h\mathbf{u}) + \eta \nabla \cdot \mathbf{u}]^2 \}. \] (131)

Tensor invariant form:

\[ V_2^k = \{(z_\alpha - \eta)u^l \frac{\partial}{\partial x^l} \left[ \frac{1}{\sqrt{g_0}} \frac{\partial}{\partial x^m}(\sqrt{g_0}h^m) \right] \}^k \]
\[ + \frac{1}{2}(z_2^2 - \eta^2)u^l \frac{\partial}{\partial x^l} \left[ \frac{1}{\sqrt{g_0}} \frac{\partial}{\partial x^m}(\sqrt{g_0}u^m) \right] \}^k \]
\[ + \frac{1}{2} \left[ \left[ \frac{1}{\sqrt{g_0}} \frac{\partial}{\partial x^l}(\sqrt{g_0}h^l) + \eta \frac{\partial}{\partial x^l}(\sqrt{g_0}u^l) \right]^2 \right] \}^k, \] (132)
2. Shore circ equations in curvilinear coordinates

1) PS99, eqn (9) depth-integrated momentum equations:

\[
\frac{\partial}{\partial t}(\vec{V} h) + \nabla_H \cdot [(\vec{V}^\alpha \vec{V}^\beta h + \int_{-h_0}^{\xi} V^\alpha_i V^\beta_i dz + Q^\alpha_i V^\beta_i (\zeta) + V^\alpha_i (\zeta) Q^\beta_i) a_\alpha a_\beta] + \frac{1}{\rho} \nabla_H \cdot T + \frac{1}{\rho} \nabla_H \cdot S + gh \nabla \zeta - \frac{1}{\rho} \tau_s + \frac{1}{\rho} \tau_B = 0
\]

2) The general form of the horizontal momentum equation PS99 (10):

\[
\frac{\partial u_H}{\partial t} + (\nabla \cdot (u u))_H = -\frac{1}{\rho} (\nabla p)_H
\]

where \( u = u_H + w \), \( ()_H \) means the horizontal components.

\[
(\nabla \cdot (u u))_H = (u^\alpha u^k)_{,k} a_\alpha
\]

\[
= \nabla_H \cdot (u_H u_H) + (u^\alpha w)_{,z} a_\alpha
\]

where \( \alpha = 1, 2 \) and \( k = 1, 2, 3 \).

Expanding \( (u^\alpha w)_{,z} \) and using coordinate transformation

\[
(u^\alpha w)_{,z} = \frac{\partial u^\alpha w}{\partial z} + \Gamma^\alpha_{\beta\gamma} u^\beta w + \Gamma^3_{\beta\gamma} u^\alpha u^3
\]

\[
= \frac{\partial u^\alpha w}{\partial z}
\]

\[
(\Gamma^\alpha_{\beta\gamma} = \Gamma^3_{\beta\gamma} = 0)
\]

\[
(\nabla \cdot (u u))_H = \nabla_H \cdot (u_H u_H) + \frac{\partial u^\alpha w}{\partial z} a_\alpha
\]
In the tensor-invariant form:

\[
\frac{\partial u^\alpha}{\partial t} + (u^\alpha u^\beta)_{,\beta} + \frac{\partial}{\partial z} (u^\alpha w) = -\frac{1}{\rho} \frac{\partial p}{\partial \xi_\beta} g^{\beta\alpha}
\]  

(140)

**Acceleration term in noninertial coordinates**

Coordinate transformation:

\[
x^i = x^i(y_1, y_2, y_3, t), \quad \tau = t
\]

(141)

or inverse transformation:

\[
y_m = y_m(x^i, x^2, x^3, \tau), \quad t = \tau
\]

(142)

Acceleration term:

\[
a = \frac{Dv^i}{D\tau} g_i - (\frac{\partial w^i}{\partial \tau} + 2 v^j w^i_{,j} - w^i w^i_{,j}) g_i
\]

(143)

where

\[
v^i = \frac{dx^i}{dt} \quad \text{— relative velocity components}
\]

\[
w^i = \frac{\partial x^i}{\partial \tau} |_{y_m} \quad \text{— frame velocity components}
\]