Brief Introduction to Tensor Algebra

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I. Basic concepts

1. different coordinate systems

a. rectangular cartesian coordinates

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2. tensor algorithm

a. contravariant and covariant components

For basis $\mathbf{g_1}, \mathbf{g_2}, \mathbf{g_3}$:

non-coplanar condition: $(\mathbf{g}_1 \times \mathbf{g}_2) \cdot \mathbf{g}_3 \neq 0$

$$\mathbf{u} = u^1\mathbf{g}_1 + u^2\mathbf{g}_2 + u^3\mathbf{g}_3 = u^i\mathbf{g}_i \tag{1}$$

For reciprocal basis $\mathbf{g^1}, \mathbf{g^2}, \mathbf{g^3}$:

$$\mathbf{g}^i \cdot \mathbf{g}_j = \delta^i_j \tag{2}$$

$$\mathbf{u} = u_1\mathbf{g^1} + u_2\mathbf{g^2} + u_3\mathbf{g^3} = u_i\mathbf{g}^i \tag{3}$$

$$\mathbf{u} \cdot \mathbf{g^j} = u^i\mathbf{g_i} \cdot \mathbf{g^j} = u^i\delta^j_i = u^j \tag{4}$$

$$\mathbf{u} \cdot \mathbf{g_j} = u^i\mathbf{g^i} \cdot \mathbf{g_j} = u_i\delta^i_j = u_j \tag{5}$$

u^i: contravariant components

u_i: covariant components

$\mathbf{g}^i$: contravariant base vectors

$\mathbf{g}_i$: covariant base vectors

b. Relationship between contravariant and covariant components

$$u^i = \mathbf{u} \cdot \mathbf{g}^i = (u_k\mathbf{g}^k) \cdot \mathbf{g}^i = u_k(\mathbf{g^k} \cdot \mathbf{g^i}) \tag{6}$$

$$u_i = \mathbf{u} \cdot \mathbf{g}_i = (u^k\mathbf{g}_k) \cdot \mathbf{g}_i = u^k(\mathbf{g_k} \cdot \mathbf{g_i}) \tag{7}$$

Introducing metric tensor:

$$g_{ik} = \mathbf{g_i} \cdot \mathbf{g_k}, \tag{8}$$

$$g^{ik} = \mathbf{g^i} \cdot \mathbf{g^k} \tag{9}$$

The relationship between contravariant and covariant components :

$$u^i = g^{ik}u_k \tag{10}$$

$$u_i = g_{ik}u^k \tag{11}$$

The relationship between contravariant and covariant base vectors:

$$\mathbf{g_i} = g_{ij}\mathbf{g^j} \tag{12}$$

$$\mathbf{g^i} = g^{ij}\mathbf{g_j} \tag{13}$$

c. Calculations of g_{ij} and g^{ij}

$$\mathbf{g_i} = \frac{\partial \mathbf{r}}{\partial x^i} \tag{14}$$

where $\mathbf{r}$ is position vector. In rectangular cartesian coordinates:

$$\mathbf{r} = y_1\mathbf{a_1} + y_2\mathbf{a_2} + y_3\mathbf{a_3} \tag{15}$$

The covariant base vectors can be expressed as :

$$\mathbf{g_i} = \frac{\partial y_1}{\partial x^i}\mathbf{a_1} + \frac{\partial y_2}{\partial x^i}\mathbf{a_2} + \frac{\partial y_3}{\partial x^i}\mathbf{a_3} \tag{16}$$

$$g_{ik} = \mathbf{g_i} \cdot \mathbf{g_k} = \frac{\partial y_j}{\partial x^i}\frac{\partial y_j}{\partial x^k} \tag{17}$$

To find g^{ij},

$$\mathbf{g^1} = \frac{\mathbf{g_2} \times \mathbf{g_3}}{\mathbf{g_1} \cdot (\mathbf{g_2} \times \mathbf{g_3})} \tag{18}$$

$$\mathbf{g^2} = \frac{\mathbf{g_3} \times \mathbf{g_1}}{\mathbf{g_1} \cdot (\mathbf{g_2} \times \mathbf{g_3})} \tag{19}$$

$$\mathbf{g^3} = \frac{\mathbf{g_1} \times \mathbf{g_2}}{\mathbf{g_1} \cdot (\mathbf{g_2} \times \mathbf{g_3})} \tag{20}$$

$$g^{ij} = (g_{rs}g_{mn} - g_{rn}g_{ms})/g \tag{21}$$

where

$$i = 1: \quad r = 2,\ m = 3; \quad j = 1: \quad s = 2,\ n = 3 \tag{22}$$

$$i = 2: \quad r = 3,\ m = 1; \quad j = 2: \quad s = 3,\ n = 1 \tag{23}$$

$$i = 3: \quad r = 1,\ m = 2; \quad j = 3: \quad s = 1,\ n = 2 \tag{24}$$

where g is the determinant of the metric tensor.

The Jacobian value of the coordinate transformation:

$$J = \begin{vmatrix} \frac{\partial y_1}{\partial x^1} & \frac{\partial y_1}{\partial x^2} & \frac{\partial y_1}{\partial x^3} \\ \frac{\partial y_2}{\partial x^1} & \frac{\partial y_2}{\partial x^2} & \frac{\partial y_2}{\partial x^3} \\ \frac{\partial y_3}{\partial x^1} & \frac{\partial y_3}{\partial x^2} & \frac{\partial y_3}{\partial x^3} \end{vmatrix} = \sqrt{g} \tag{25}$$

another expression of g^{ik}

$$g^{ik} = \mathbf{g^i} \cdot \mathbf{g^k} = \frac{\partial x^i}{\partial y_k} \frac{\partial x^j}{\partial y_k} \tag{26}$$

Some examples:

Rectangular cartesian coordinates,

$g_{ij} = \delta_{ij}$.

Cylindrical polar coordinates,

$g_{11} = 1, g_{22} = \rho^2, g_{33} = 1$, and $g_{12} = g_{23} = g_{31} = 0$.

Orthogonal curvilinear coordinates,

$g_{ij} = 0, i \neq j$.

d. Dot products in general coordinates

1) Scalar products of two vectors

In generalized coordinates

$$\mathbf{u} \cdot \mathbf{v} = (u^i \mathbf{g_i}) \cdot (v_j \mathbf{g^j}) \tag{27}$$

according to

$$\mathbf{g_i} \cdot \mathbf{g^j} = \delta_i^j \tag{28}$$

we obtain:

$$\mathbf{u} \cdot \mathbf{v} = u^i v_i \tag{29}$$

2) dot products of second order tensors

$$\mathbf{T} = T_{ij}\mathbf{g^i}\mathbf{g^j} \quad \mathbf{S} = S^{kl}\mathbf{g_k}\mathbf{g_l} \tag{30}$$

The dot products:

$$\begin{aligned}
\mathbf{T} \cdot \mathbf{S} &= (T_{ij}\mathbf{g^i}\mathbf{g^j}) \cdot (S^{kl}\mathbf{g_k}\mathbf{g_l}) && (31)\\
&= T_{ij}\mathbf{g^i}\delta_k^j S^{kl}\mathbf{g_l} && (32)\\
&= T_{ij}S^{jl}\mathbf{g^i}\mathbf{g_l} && (33)
\end{aligned}$$

3) double dot products of second order tensors

The defination of double dot product:

$$(\mathbf{ab}) : (\mathbf{cd}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) \tag{34}$$

$$\begin{aligned}
\mathbf{T} : \mathbf{S} &= (T_{ij}\mathbf{g^i}\mathbf{g^j}) : (S^{kl}\mathbf{g_k}\mathbf{g_l}) && (35)\\
&= T_{ij}S^{kl}(\mathbf{g^i} \cdot \mathbf{g_k})(\mathbf{g^j} \cdot \mathbf{g_l}) && (36)\\
&= T_{ij}S^{ij} && (37)
\end{aligned}$$

II. Differentiation of tensors

1.The differentiation of base vectors

1) derivatives of base vectors

$$\frac{\partial \mathbf{g_i}}{\partial x^j} = \Gamma_{ij}^1\mathbf{g_1} + \Gamma_{ij}^2\mathbf{g_2} + \Gamma_{ij}^3\mathbf{g_3} = \Gamma_{ij}^k\mathbf{g_k} \tag{38}$$

– Γ_{ij}^k Christoffel symbols of the second kind.

$$\Gamma_{ij}^k\mathbf{g_k} \cdot \mathbf{g^l} = \Gamma_{ij}^k\delta_k^l = \frac{\partial \mathbf{g_i}}{\partial x^j} \cdot \mathbf{g^l} \tag{39}$$

$$\Gamma_{ij}^k = \frac{\partial \mathbf{g_i}}{\partial x^j} \cdot \mathbf{g^k} \tag{40}$$

Christoffel symbols of the first kind:

$$\frac{\partial \mathbf{g_i}}{\partial x^j} = \Gamma_{ij1}\mathbf{g^1} + \Gamma_{ij2}\mathbf{g^2} + \Gamma_{ij3}\mathbf{g^3} = \Gamma_{ijk}\mathbf{g^k} \tag{41}$$

Formulas for the Christoffel symbols of the first kind are

$$\Gamma_{ijk} = \frac{\partial \mathbf{g_i}}{\partial x^j} \cdot \mathbf{g_k} \tag{42}$$

Calculation of Christoffel symbols

$$\frac{\partial \mathbf{g_i}}{\partial x^j} = \frac{\partial}{\partial x^j}\frac{\partial \mathbf{r}}{\partial x^i} = \frac{\partial^2 \mathbf{r}}{\partial x^j \partial x^i} \tag{43}$$

and

$$\mathbf{g^k} = \frac{\partial x^k}{\partial y_l} \tag{44}$$

$$\Gamma^k_{ij} = \frac{\partial^2 y_l}{\partial x^i \partial x^j}\frac{\partial x^k}{\partial y_l} \tag{45}$$

and

$$\Gamma_{ijk} = \frac{\partial^2 y_l}{\partial x^i \partial x^j}\frac{\partial y_l}{\partial x^k} \tag{46}$$

2) The relationship between the Christoffel symbols

$$\Gamma^k_{ij} = \frac{\partial \mathbf{g_i}}{\partial x^j} \cdot \mathbf{g^k} = \frac{\partial \mathbf{g_i}}{\partial x^j} \cdot g^{kl}\mathbf{g_l} = g^{kl}\Gamma_{ijl} \tag{47}$$

similarly,

$$\Gamma_{ijk} = g_{kl}\Gamma^l_{ij} \tag{48}$$

$\frac{\partial \mathbf{g^j}}{\partial x^k}$?

$$\mathbf{g_i} \cdot \mathbf{g^j} = \delta^j_i \tag{49}$$

$$\frac{\partial \mathbf{g_i}}{\partial x^k} \cdot \mathbf{g^j} + \mathbf{g_i} \cdot \frac{\partial g^j}{\partial x^k} = 0 \tag{50}$$

$$\Gamma^j_{ik} = -\mathbf{g_i} \cdot \frac{\partial \mathbf{g^j}}{\partial x^k} \tag{51}$$

$$\frac{\partial \mathbf{g^j}}{\partial x^k} = -\Gamma^j_{ik}\mathbf{g^i} \tag{52}$$

3) Examples of Christoffel symbols:

(1) Cartesian coordinates (rectangular or generlized cartesian)

$$\Gamma^k_{ij} = \Gamma_{ijk} = 0 \tag{53}$$

(2) Cylindrical polars

In cylindrical polar, all the components of Γ^k_{ij} are zero except for

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{\rho}, \quad \Gamma^1_{22} = -\rho \tag{54}$$

or the Christoffel symbol of the first kind,

$$\Gamma_{122} = \Gamma_{212} = \rho, \quad \Gamma_{221} = -\rho \tag{55}$$

(3)Spherical polars

non-zero components of Christoffel symbols for spherical polar coordinates are

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r}, \quad \Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r} \tag{56}$$

$$\Gamma^1_{22} = -r, \quad \Gamma^3_{23} = \Gamma^3_{32} = cos\theta, \quad \Gamma^1_{33} = -rsin^2\theta, \quad \Gamma^2_{33} = -sin\theta cos\theta \tag{57}$$

2. Covariant derivatives of vector fields

$$\frac{\partial \mathbf{u}}{\partial x^j} = u^i\frac{\partial \mathbf{g_i}}{\partial x^j} + \frac{\partial u^i}{\partial x^j}\mathbf{g_i} \tag{58}$$

$$= u^i\Gamma^k_{ij}\mathbf{g_k} + \frac{\partial u^i}{\partial x^j}\mathbf{g_i} = (\frac{\partial u^i}{\partial x^j} + \Gamma^i_{kj}u^k)\mathbf{g_i} \tag{59}$$

The covariant derivative of the contravariant vector u^i.

$$u^i_{,j} = \frac{\partial u^i}{\partial x^j} + \Gamma^i_{kj}u^k \tag{60}$$

$$\frac{\partial \mathbf{u}}{\partial x^j} = u_i\frac{\partial \mathbf{g^i}}{\partial x^j} + \frac{\partial u_i}{\partial x^j}\mathbf{g^i} \tag{61}$$

$$= -u_i\Gamma^i_{kj}\mathbf{g^k} + \frac{\partial u_i}{\partial x^j}\mathbf{g^i} = (\frac{\partial u_i}{\partial x^j} - \Gamma^k_{ij}u_k)\mathbf{g^i} \tag{62}$$

The covariant derivative of the covariant vector u_i

$$u_{i,j} = \frac{\partial u_i}{\partial x^j} - \Gamma^k_{ij} u_k \tag{63}$$

The relationship between these two kinds of covariant derivatives:

$$\frac{\partial \mathbf{u}}{\partial x^j} = u_{i,j}\mathbf{g^i} = u^i_{,j}\mathbf{g_i} = u^i_{,j} g_{ik}\mathbf{g^k} = u^k_{,j} g_{ki}\mathbf{g^i} \tag{64}$$

So

$$u_{i,j} = g_{ik} u^k_{,j} \tag{65}$$

3. Grad, div and curl

1) gradient of a scalar

The grad operator

$$grad = \nabla = \mathbf{g^i}\frac{\partial}{\partial x^i} \tag{66}$$

The gradient of a scalar:

$$grad\phi = \nabla\phi = \mathbf{g^i}\frac{\partial \phi}{\partial x^i} \tag{67}$$

$$grad\phi = \mathbf{g^i}\frac{\partial \phi}{\partial x^i} = g^{ij}\mathbf{g_j}\frac{\partial \phi}{\partial x^i} \tag{68}$$

2) gradient of a vector

$$grad\mathbf{u} = \frac{\partial \mathbf{u}}{\partial x^i}\mathbf{g}^i \tag{69}$$

$$\frac{\partial \mathbf{u}}{\partial x^i} = (\frac{\partial u^k}{\partial x^i} + u^j \Gamma^k_{ij})\mathbf{g}_k = u^k_{,i}\mathbf{g}_k \tag{70}$$

$$grad\mathbf{u} = u^k_{,i}\mathbf{g_k}\mathbf{g^i} \tag{71}$$

$$\frac{\partial \mathbf{u}}{\partial x^i} = u_{k,i}\mathbf{g^k} \tag{72}$$

$$grad\mathbf{u} = v_{k,i}\mathbf{g^k}\mathbf{g^i} \tag{73}$$

3) divergence of a vector

$$div\mathbf{u} = \frac{\partial \mathbf{u}}{\partial x^i} \cdot \mathbf{g}^i \tag{74}$$

from

$$\frac{\partial \mathbf{u}}{\partial x^i} = (\frac{\partial u^k}{\partial x^i} + u^j\Gamma^k_{ij})\mathbf{g}_k = u^k_{,i}\mathbf{g}_k \tag{75}$$

$$\begin{aligned} div\mathbf{u} &= \frac{\partial \mathbf{u}}{\partial x^i} \cdot \mathbf{g}^i && (76)\\ &= u^k_{,i}\mathbf{g}_k \cdot \mathbf{g}^i && (77)\\ &= u^i_{,i} && (78) \end{aligned}$$

$$u^i_{,i} = \frac{\partial u^i}{\partial x^i} + u^j\Gamma^i_{ij} \tag{79}$$

$$\Gamma^i_{ij} = \frac{\partial ln(\sqrt{g})}{\partial x^j} \tag{80}$$

so

$$\begin{aligned} u^i_{,i} &= \frac{\partial u^i}{\partial x^i} + u^j\frac{\partial ln(\sqrt{g})}{\partial x^j} && (81)\\ &= \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}(\sqrt{g}u^i) && (82) \end{aligned}$$

$$div\mathbf{u} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}(\sqrt{g}u^i) \tag{83}$$

4) divergence of a second-order tensor

A second order tensor can be written as

$$\mathbf{T} = T^{ij}\mathbf{g_i}\mathbf{g_j} \tag{84}$$

$$\frac{\partial \mathbf{T}}{\partial x^k} = \frac{\partial T^{ij}\mathbf{g_i}}{\partial x^k}\mathbf{g_j} + T^{ij}\mathbf{g_i}\Gamma^m_{jk}\mathbf{g_m} \tag{85}$$

$$= \frac{\partial T^{ij}}{\partial x^k}\mathbf{g_i}\mathbf{g_j} + T^{ij}\Gamma^m_{ik}\mathbf{g_m}\mathbf{g_j} + T^{ij}\mathbf{g_i}\Gamma^m_{jk}\mathbf{g_m} \tag{86}$$

$$\frac{\partial \mathbf{T}}{\partial x^k}\cdot\mathbf{g^k} = \frac{\partial T^{ik}}{\partial x^k}\mathbf{g_i} + T^{ik}\Gamma^m_{ik}\mathbf{g_m} + T^{ij}\mathbf{g_i}\Gamma^k_{jk} \tag{87}$$

$$= \frac{\partial T^{ik}}{\partial x^k}\mathbf{g_i} + T^{jk}\Gamma^i_{jk}\mathbf{g_i} + T^{ij}\Gamma^k_{jk}\mathbf{g_i} \tag{88}$$

$$div\mathbf{T} = T^{ik}_{,k}\mathbf{g_i} \tag{89}$$

$$\mathbf{T} = T_{ij}\mathbf{g^i}\mathbf{g^j} \tag{90}$$

$$div\mathbf{T} = g^{jk}T_{ij,k}\mathbf{g^i} \tag{91}$$

$$\mathbf{T} = \mathbf{T^i_j}\mathbf{g_i}\mathbf{g^j} \tag{92}$$

$$div\mathbf{T} = g^{jk}T^i_{j,k}\mathbf{g_i} \tag{93}$$

where

$$T^{ij}_{,k} = \frac{\partial T^{ij}}{\partial x^k} + \Gamma^i_{mk}T^{mj} + \Gamma^j_{mk}T^{im} \tag{94}$$

$$T_{ij,k} = \frac{\partial T_{ij}}{\partial x^k} - \Gamma^m_{ik}T_{mj} - \Gamma^m_{jk}T_{im} \tag{95}$$

$$T^i_{j,k} = \frac{\partial T^i_j}{\partial x^k} + \Gamma^i_{mk} - \Gamma^m_{jk}T^i_m \tag{96}$$

5) curl of vectors

$$curl\mathbf{u} = \mathbf{g}^i \times \frac{\partial \mathbf{u}}{\partial x^i} \tag{97}$$

$$curl\mathbf{u} = \frac{1}{\sqrt{g}}e^{ijk}u_{k,j}\mathbf{g_i} \tag{98}$$

where e^{ijk} are the permutation symbols.

$$e_{ijk} = \begin{cases} 0 & \text{if any of } i, j, k \text{ are the same} \\ +1 & \text{if } (i,j,k) \text{ is a cyclic permutation of (1,2,3)} \\ -1 & \text{otherwise} \end{cases} \tag{99}$$

III. Tensor expression of basic terms in fluid mechanics

1. convective terms

The acceleration term:

$$\begin{aligned}
\mathbf{a} &= \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot grad)\mathbf{u} && (100)\\
&= \frac{\partial \mathbf{u}}{\partial t} + (grad\mathbf{u}) \cdot \mathbf{u} && (101)\\
&= \frac{\partial \mathbf{u}}{\partial t} + div(\mathbf{uu}) - \mathbf{u}(div\mathbf{u}) && (102)\\
&= \frac{\partial \mathbf{u}}{\partial t} + grad(\frac{1}{2}\mathbf{u} \cdot \mathbf{u}) + \omega \times \mathbf{u} && (103)
\end{aligned}$$

1) $(\mathbf{u} \cdot grad)\mathbf{u}$

$$\mathbf{u} = u^i \mathbf{g_i} \tag{104}$$

$$grad = \mathbf{g}^i \frac{\partial}{\partial x^i} \tag{105}$$

$$\begin{aligned}
(\mathbf{u} \cdot grad)\mathbf{u} &= u^i \frac{\partial \mathbf{u}}{\partial x^i} && (106)\\
&= u^i u^j_{,i} \mathbf{g_j} && (107)
\end{aligned}$$

2) $div(\mathbf{uu})$

$$\mathbf{uu} = u^i u^j \mathbf{g_i g_j} \tag{108}$$

$$div(\mathbf{uu}) = (u^i u^k)_{,k} \mathbf{g_i} \tag{109}$$

3) $grad(\frac{1}{2}\mathbf{u} \cdot \mathbf{u})$

$$grad(\frac{1}{2}\mathbf{u} \cdot \mathbf{u}) = \frac{1}{2} u^i u_i = \frac{1}{2} g_{ij} u^i u^j \tag{110}$$

2. pressure gradient term

$$\nabla\eta = \frac{\partial\eta}{\partial x^i}\mathbf{g^i} \tag{111}$$

$$= \frac{\partial\eta}{\partial x^i}g^{ij}\mathbf{g_j} = \eta!^{\mathbf{j}}\mathbf{g_j} \tag{112}$$

where $\eta!^j$ is the contravariant component of the gradient.

3. diffusion terms

$$div\mathbf{T} = div(\nu\mathbf{D}) \tag{113}$$

D is the rate-of-strain tensor:

$$\mathbf{D} = \frac{1}{2}[grad\mathbf{u} + (grad\mathbf{u})^T] \tag{114}$$

$(grad\mathbf{u})^T$ is the transpose of $grad\mathbf{u}$.

$$div\mathbf{T} = \frac{1}{\sqrt{g_0}}\frac{\partial\sqrt{g_0}\nu D^{ik}}{\partial x^k}\mathbf{a}_i + \nu D^{jk}\Gamma^i_{jk}\mathbf{a}_i \tag{115}$$

Smagorinsky's subgrid turbulence

In rectangular cartesian coordinates:

$$\nu_s = c\Delta x\Delta y\frac{1}{2}|grad\mathbf{u} + (grad\mathbf{u})^T| \tag{116}$$

In curvilinear coordinates:

$$\begin{aligned} \nu_s &= c\sqrt{g_0}\Delta\xi_1\Delta\xi_2|\mathbf{D}| \\ &= c\sqrt{g_0}\Delta\xi_1\Delta\xi_2(\mathbf{D}:\mathbf{D})^{1/2} \\ &= c\sqrt{g_0}\Delta\xi_1\Delta\xi_2(D^{ij}D_{ij})^{1/2} \end{aligned} \tag{117}$$

IV. Two examples of coordinate transformations

1. Boussinesq equations in curvilinear coordinates

Boussinesq equations in rectangular cartesian coordinates:

$$\eta_t + \nabla\cdot\mathbf{M} = 0, \tag{118}$$

$$\begin{aligned}\mathbf{M} &= (h+\eta)\mathbf{u} + (h+\eta)[\frac{z_\alpha^2}{2} - \frac{1}{6}(h^2 - h\eta + \eta^2)]\nabla(\nabla\cdot\mathbf{u}) \\ &\quad +(h+\eta)[z_\alpha + \frac{1}{2}(h-\eta)]\nabla[\nabla\cdot(h\mathbf{u})],\end{aligned} \tag{119}$$

$$\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} + g\nabla\eta + \mathbf{V_1} + \ \mathbf{V_2} = 0, \tag{120}$$

$$\mathbf{V_1} = \frac{z_\alpha^2}{2}\nabla(\nabla\cdot\mathbf{u}_t) + z_\alpha\nabla[\nabla\cdot(h\mathbf{u}_t)] - \nabla[\frac{1}{2}\eta^2\nabla\cdot\mathbf{u}_t + \eta\nabla\cdot(h\mathbf{u}_t)], \tag{121}$$

$$\begin{aligned}\mathbf{V_2} &= \nabla\{(z_\alpha - \eta)(\mathbf{u}\cdot\nabla)[\nabla\cdot(h\mathbf{u})] + \frac{1}{2}(z_\alpha^2 - \eta^2)(\mathbf{u}\cdot\nabla)(\nabla\cdot\mathbf{u})\} \\ &\quad +\frac{1}{2}\nabla\{[\nabla\cdot(h\mathbf{u}) + \eta\nabla\cdot\mathbf{u}]^2\}.\end{aligned} \tag{122}$$

1)transformation of mass equation

$$\eta_t + \nabla\cdot\mathbf{M} = 0, \tag{123}$$

where $\mathbf{M}$ is the depth-integrated volume flux given by

$$\begin{aligned}\mathbf{M} &= (h+\eta)\mathbf{u} + (h+\eta)[\frac{z_\alpha^2}{2} - \frac{1}{6}(h^2 - h\eta + \eta^2)]\nabla(\nabla\cdot\mathbf{u}) \\ &\quad +(h+\eta)[z_\alpha + \frac{1}{2}(h-\eta)]\nabla[\nabla\cdot(h\mathbf{u})],\end{aligned} \tag{124}$$

Tensor invariant forms:

$$\eta_t + \frac{1}{\sqrt{g_0}}\frac{\partial}{\partial x^k}(\sqrt{g_0}M^k) = 0, \tag{125}$$

$$
\begin{aligned}
M^k &= (h+\eta)u^k + (h+\eta)[\frac{z_\alpha^2}{2} - \frac{1}{6}(h^2 - h\eta + \eta^2)][\frac{1}{\sqrt{g_0}}\frac{\partial}{\partial x^l}(\sqrt{g_0}u^l)]!^k \\
&+(h+\eta)[z_\alpha + \frac{1}{2}(h-\eta)][\frac{1}{\sqrt{g_0}}\frac{\partial}{\partial x^l}(\sqrt{g_0}hu^l)]!^k. \qquad (126)
\end{aligned}
$$

2)transformation of momentum equations

$$
\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} + g\nabla\eta + \mathbf{V_1} + \mathbf{V_2} = 0, \qquad (127)
$$

The tensor invariant form:

$$
\frac{\partial u^k}{\partial t} + g\eta!^k + u^l u^k_{,l} + V_1^k + V_2^k = 0, \qquad (128)
$$

$$
\mathbf{V_1} = \frac{z_\alpha^2}{2}\nabla(\nabla\cdot\mathbf{u}_t) + z_\alpha\nabla[\nabla\cdot(h\mathbf{u}_t)] - \nabla[\frac{1}{2}\eta^2\nabla\cdot\mathbf{u}_t + \eta\nabla\cdot(h\mathbf{u}_t)], \qquad (129)
$$

Tensor invariant form:

$$
\begin{aligned}
V_1^k &= \frac{z_\alpha^2}{2}[\frac{1}{\sqrt{g_0}}\frac{\partial}{\partial x^l}(\sqrt{g_0}u_t^l)]!^k \\
&+z_\alpha[\frac{1}{\sqrt{g_0}}\frac{\partial}{\partial x^l}(\sqrt{g_0}hu_t^l)]!^k \\
&-[\frac{\eta^2}{2\sqrt{g_0}}\frac{\partial}{\partial x^l}(\sqrt{g_0}u_t^l) + \frac{\eta}{\sqrt{g_0}}\frac{\partial}{\partial x^l}(\sqrt{g_0}hu_t^l)]!^k, \qquad (130)
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{V_2} &= \nabla\{(z_\alpha-\eta)(\mathbf{u}\cdot\nabla)[\nabla\cdot(h\mathbf{u})] + \frac{1}{2}(z_\alpha^2-\eta^2)(\mathbf{u}\cdot\nabla)(\nabla\cdot\mathbf{u})\} \\
&+\frac{1}{2}\nabla\{[\nabla\cdot(h\mathbf{u}) + \eta\nabla\cdot\mathbf{u}]^2\}. \qquad (131)
\end{aligned}
$$

Tensor invariant form:

$$
\begin{aligned}
V_2^k &= \{(z_\alpha-\eta)u^l\frac{\partial}{\partial x^l}[\frac{1}{\sqrt{g_0}}\frac{\partial}{\partial x^m}(\sqrt{g_0}hu^m)]\}!^k \\
&+\frac{1}{2}(z_\alpha^2-\eta^2)u^l\frac{\partial}{\partial x^l}[\frac{1}{\sqrt{g_0}}\frac{\partial}{\partial x^m}(\sqrt{g_0}u^m)]\}!^k \\
&+\frac{1}{2}\{[\frac{1}{\sqrt{g_0}}\frac{\partial}{\partial x^l}(\sqrt{g_0}hu^l) + \frac{\eta}{\sqrt{g_0}}\frac{\partial}{\partial x^l}(\sqrt{g_0}u^l)]^2\}!^k, \qquad (132)
\end{aligned}
$$

2.Shorecirc equations in curvilinear coordinates

1) PS99, eqn (9) depth-integrated momentum equations:

$$\begin{aligned}\frac{\partial}{\partial t}(\tilde{\mathbf{V}}h) &+ \nabla_H \cdot [(\tilde{V}^\alpha \tilde{V}^\beta h + \int_{-h_0}^{\overline{\zeta}} V_1^\alpha V_1^\beta dz + Q_w^\alpha V_1^\beta(\overline{\zeta}) + V_1^\alpha(\overline{\zeta}) Q_w^\beta)\mathbf{a}_\alpha \mathbf{a}_\beta] \\ &+ \frac{1}{\rho}\nabla_H \cdot \mathbf{T} + \frac{1}{\rho}\nabla_H \cdot \mathbf{S} + gh\nabla\zeta - \frac{1}{\rho}\tau_{\mathbf{s}} + \frac{1}{\rho}\tau_{\mathbf{B}} = 0\end{aligned} \tag{133}$$

tensor-invariant forms

$$\begin{aligned}&\frac{\partial}{\partial t}(\tilde{V}^\alpha h) + (\tilde{V}^\alpha \tilde{V}^\beta h + \int_{-h_0}^{\overline{\zeta}} V_1^\alpha V_1^\beta dz + Q_w^\alpha V_1^\beta(\overline{\zeta}) + V_1^\alpha(\overline{\zeta}) Q_w^\beta)_{,\beta} \\ &\qquad + \frac{1}{\rho}T^{\alpha\beta}_{,\beta} + \frac{1}{\rho}S^{\alpha\beta}_{,\beta} + gh\frac{\partial \zeta}{\partial \xi_\beta} g^{\beta\alpha} - \frac{1}{\rho}\tau_s^\alpha + \frac{1}{\rho}\tau_B^\alpha = 0\end{aligned} \tag{134}$$

2)The general form of the horizontal momentum equation PS99 (10):

$$\frac{\partial \mathbf{u_H}}{\partial t} + (\nabla \cdot (\mathbf{uu}))_H = -\frac{1}{\rho}(\nabla p)_H \tag{135}$$

where $\mathbf{u} = \mathbf{u_H} + \mathbf{w}$, $()_H$ means the horizontal components.

$$\begin{aligned}(\nabla \cdot (\mathbf{uu}))_H &= (u^\alpha u^k)_{,k}\mathbf{a}_\alpha \\ &= \nabla_H \cdot (\mathbf{u_H u_H}) + (u^\alpha w)_{,z}\mathbf{a}_\alpha\end{aligned} \tag{136}$$

where $\alpha = 1, 2$ and $k = 1, 2, 3$.

Expending $(u^\alpha w)_{,z}$ and using coordinate transformation

$$\begin{aligned}(u^\alpha w)_{,z} &= \frac{\partial u^\alpha w}{\partial z} + \Gamma^\alpha_{\beta 3} u^\beta w + \Gamma^3_{\beta 3} u^\alpha u^\beta \\ &= \frac{\partial u^\alpha w}{\partial z}\end{aligned} \tag{137}$$

$$(\Gamma^\alpha_{\beta 3} = \Gamma^3_{\beta 3} = 0) \tag{138}$$

$$(\nabla \cdot (\mathbf{uu}))_H = \nabla_H \cdot (\mathbf{u_H u_H}) + \frac{\partial u^\alpha w}{\partial z}\mathbf{a}_\alpha \tag{139}$$

In the tensor-invariant form:

$$\frac{\partial u^{\alpha}}{\partial t} + (u^{\alpha} u^{\beta})_{,\beta} + \frac{\partial}{\partial z}(u^{\alpha} w) = -\frac{1}{\rho}\frac{\partial p}{\partial \xi_{\beta}} g^{\beta\alpha} \tag{140}$$

Acceleration term in noninertial coordinates

Coordinate transformation:

$$x^i = x^i(y_1, y_2, y_3, t), \qquad \tau = t \tag{141}$$

or inverse transformation:

$$y_m = y_m(x^i, x^2, x^3, \tau), \qquad t = \tau \tag{142}$$

Acceleration term:

$$\mathbf{a} = \frac{Dv^i}{D\tau}\mathbf{g}_i - (\frac{\partial w^i}{\partial \tau} + 2v^j w^i_{,j} - w^j w^i_{,j})\mathbf{g}_i \tag{143}$$

where

$v^i = \frac{dx^i}{dt}$ — relative velocity components

$w^i = \frac{\partial x^i}{\partial t}|_{y_m}$ — frame velocity components