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Shear Dispersion of Momentum in the Nearshore

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Abstract

It is shown that the vertical nonuniformity of the short-wave-averaged horizontal velocities leads to mixing-like terms for the horizontal velocity in the depth-integrated equations of momentum. The mechanism is analogous to Taylor's (1953, 1954) shear-dispersion mechanism for solutes in a shear flow. The results presented here are an extension of the results found by Svendsen & Putrevu (1994) to the general case of unsteady flow over an arbitrary bottom topography.

1 Introduction

In a previous paper (Svendsen & Putrevu 1994, SP94 hereafter), we considered the case of steady longshore currents on an alongshore-uniform coast and found that the vertical nonuniformity of the currents leads to a mixing-like term in the depth-integrated alongshore momentum equation. The mechanism by which this happens is analogous to the shear-dispersion mechanism found by Taylor (1953, 1954) for the lateral spreading of solutes in a shear flow.

For the case considered in SP94, the lateral mixing caused by the shear-dispersion mechanism was an order of magnitude larger than the turbulent lateral mixing, even inside the surf zone (*e.g.*, Figure 1). This result suggests that the dispersive mixing will be a major contributor to the total lateral mixing in the nearshore region. Therefore, it is desirable to extend the results of SP94 to the general case of unsteady flow over an arbitrary bottom topography. This paper presents such an extension.

Recently, Smith (1997) presented a rather general derivation of the shear dispersion of momentum using a multi-mode representation of the flow. The principal difference between

our work and that of Smith is that we specifically concentrate on nearshore flows driven by short-waves. Unlike in Smith's case, the short-wave-induced volume flux is an important component of the flows we are interested in. (For a zero wave-induced volume flux, our results are similar to those of Smith.) Another (less important) difference is that our style of calculation is quite different from that of Smith and leads to a result that is somewhat more general than that given by Smith even for the case in which there is no wave-induced volume flux. However, it is also shown that the extra terms we obtain (relative to Smith's results) are likely to be small in most situations. Finally, in Smith's derivation special attention is needed to allow for a no-slip condition at the bed. Here we instead allow for a slip velocity at the bed. However, the solution can, in principle, be readily extended to incorporate the no-slip condition at the bed.

The present paper is organized as follows. Section 2 discusses the depth-integrated, short-wave-averaged equations of continuity and momentum for the case in which the short-wave-averaged horizontal velocities are allowed to vary with the vertical coordinate. The evaluation of the extra terms that arise from the vertical nonuniformity of the short-wave-averaged horizontal velocities forms the subject of Sections 3 and 4. Section 5 discusses the implications of the results derived in Section 4. The final section is devoted to a few concluding remarks.

2 Depth-integrated, short-wave-averaged equations

We start with the depth-integrated, short-wave-averaged equations of continuity and horizontal momentum which allow for the short-wave-averaged velocities to vary with the vertical. These equations are derived following the steps given in Phillips (1977) or Mei (1989) and are minor extensions of the equations given therein [eqs. 3.6.4 and 3.6.11 of Phillips (1977, pp. 61–62) and eqs. 2.50 and 2.51 of Mei (1989, p. 463)]. (Phillips and Mei only considered situations where the short-wave-averaged velocities are uniform over the vertical.)

Following Phillips (1977)¹ we split the instantaneous horizontal velocity into four components

$$u_\alpha = u'_\alpha + u_{w\alpha} + \tilde{V}_\alpha + V_{1\alpha} \quad (1)$$

where u'_α is the turbulence component, $u_{w\alpha}$ is the wave component (whose short-wave-averaged value is zero below trough level), and \tilde{V}_α and $V_{1\alpha}$ are two components of the short-wave-averaged velocity. The first component, \tilde{V}_α (\tilde{U}_α in Phillips' notation), is uniform over depth

¹There are some differences in the way Phillips and Mei define their variables [see Svendsen & Putrevu (1996) for a discussion].

and is given by

$$\tilde{V}_\alpha = \frac{1}{h} \overline{\int_{-h_0}^{\zeta} u_\alpha dz} \quad (2)$$

where z is the vertical coordinate (measured from the still water level). In the above, an overbar denotes averaging over a short-wave period, h_0 , ζ , and ζ_t represent the still water depth, instantaneous water surface elevation, and the elevation of the wave trough level, respectively and $h = h_0 + \bar{\zeta}$ represents the total depth.

The second component of the short-wave-averaged velocity, $V_{1\alpha}$, accounts for the vertical variation and satisfies

$$\overline{\int_{-h_0}^{\zeta} V_{1\alpha} dz} = -\overline{\int_{\zeta_t}^{\zeta} u_{w\alpha} dz} = -Q_{w\alpha} \quad (3)$$

where $Q_{w\alpha}$ is short-wave-induced volume flux. (In terms of Phillips' variables, this component is analogous to $\tilde{U}_\alpha - U_\alpha$.) Notice that, in addition to representing the vertical variation of V_α , the depth-averaged value of $V_{1\alpha}$ is the part of the short-wave-averaged motion that compensates for the volume flux due to the short wave motion. Figure 2 shows the definitions of the two components of the short-wave-averaged velocity.

In terms of these variables, the depth-integrated, short-wave-averaged equations read

$$\frac{\partial \bar{\zeta}}{\partial t} + \frac{\partial}{\partial x_\alpha} (\tilde{V}_\alpha h) = 0 \quad (4)$$

and

$$\begin{aligned} \frac{\partial}{\partial t} (\tilde{V}_\alpha h) + \frac{\partial}{\partial x_\beta} \left[\tilde{V}_\alpha \tilde{V}_\beta h + \overline{\int_{-h_0}^{\zeta} V_{1\alpha} V_{1\beta} dz} + \overline{\int_{\zeta_t}^{\zeta} (u_{w\alpha} V_{1\beta} + V_{1\alpha} u_{w\beta}) dz} \right] \\ + \frac{1}{\rho} \frac{\partial T_{\alpha\beta}}{\partial x_\beta} + \frac{1}{\rho} \frac{\partial S_{\alpha\beta}}{\partial x_\beta} + gh \frac{\partial \bar{\zeta}}{\partial x_\alpha} - \frac{\tau_\alpha^S}{\rho} + \frac{\tau_\alpha^B}{\rho} = 0 \end{aligned} \quad (5)$$

where τ_α^S , τ_α^B , $S_{\alpha\beta}$, and $T_{\alpha\beta}$ are the surface shear stress, the bottom shear stress, the short-wave-induced radiation stress, and the depth-integrated Reynolds' stress, respectively. In (5) the radiation stress is defined by

$$S_{\alpha\beta} = \overline{\int_{-h_0}^{\zeta} (\rho u_{w\alpha} u_{w\beta} + p \delta_{\alpha\beta}) dz} - \frac{1}{2} \rho g h^2 \delta_{\alpha\beta} \quad (6)$$

where p is the total pressure and $\delta_{\alpha\beta}$ is the Kronecker delta function. This definition of the radiation stress is slightly different from the one used by Phillips (1977, eq. 3.6.12, p. 62).

Note that since we have allowed for a time variation in (4) and (5), these equations model both mean currents and long waves.

In this work, we concentrate on nearshore flows whose main driving force is the gradient of the radiation stress. We specifically exclude wind-driven flows. Thus, we will neglect surface stresses in the following. Furthermore, we assume that the first integral in (3) may be simplified as follows

$$\overline{\int_{-h_0}^{\zeta} V_{1\alpha} dz} \approx \int_{-h_0}^{\bar{\zeta}} V_{1\alpha} dz = -Q_{w\alpha} \quad (7)$$

which implies that we assume that $V_{1\alpha}$ is approximately constant in the interval $\bar{\zeta}$ to ζ . Since we are specifically excluding wind-driven flows, this is a reasonable approximation that does not change the characteristics of the results. (7) also implies that the integrals involving V_1 in (5) may be written as

$$\overline{\int_{-h_0}^{\zeta} V_{1\alpha} V_{1\beta} dz} + \overline{\int_{\zeta_t}^{\zeta} (u_{w\alpha} V_{1\beta} + V_{1\alpha} u_{w\beta}) dz} \approx \int_{-h_0}^{\bar{\zeta}} V_{1\alpha} V_{1\beta} dz + V_{1\beta}(\bar{\zeta}) Q_{w\alpha} + V_{1\alpha}(\bar{\zeta}) Q_{w\beta} \quad (8)$$

Thus, in the following, we will consider the following simplified form of the momentum equation

$$\begin{aligned} \frac{\partial}{\partial t} (\tilde{V}_\alpha h) + \frac{\partial}{\partial x_\beta} \left[\tilde{V}_\alpha \tilde{V}_\beta h + \int_{-h_0}^{\bar{\zeta}} V_{1\alpha} V_{1\beta} dz + V_{1\beta}(\bar{\zeta}) Q_{w\alpha} + V_{1\alpha}(\bar{\zeta}) Q_{w\beta} \right] \\ + \frac{1}{\rho} \frac{\partial T_{\alpha\beta}}{\partial x_\beta} + \frac{1}{\rho} \frac{\partial S_{\alpha\beta}}{\partial x_\beta} + gh \frac{\partial \bar{\zeta}}{\partial x_\alpha} + \frac{\tau_\alpha^B}{\rho} = 0 \end{aligned} \quad (9)$$

The terms involving V_1 in (9) represent the effects of the vertical nonuniformity of the short-wave-averaged velocities, and it is these terms that give rise to the dispersive mixing. The goal of the following is to express the dispersive terms in terms of \tilde{V}_α and the short-wave-related quantities ($S_{\alpha\beta}$, $Q_{w\alpha}$, $u_{w\alpha}$, etc.) so that (4) and (9) reduce to equations in which the only unknowns are $\bar{\zeta}$ and \tilde{V}_α . The benefit of reducing (4) and (9) to equations in which the only unknowns are $\bar{\zeta}$ and \tilde{V}_α is that by doing so we can incorporate the effects of three-dimensionality in two dimensional calculations.

The first step in the calculation of the dispersive terms is the determination of the vertical structure of $V_{1\alpha}$. The next section discusses the determination of the vertical structure of $V_{1\alpha}$.

3 Vertical structure of $V_{1\alpha}$

To derive the equation governing the vertical structure of $V_{1\alpha}$, we start with the horizontal momentum equation

$$\frac{\partial u_\alpha}{\partial t} + \frac{\partial}{\partial x_\beta} (u_\alpha u_\beta) + \frac{\partial}{\partial z} (u_\alpha w) = \frac{-1}{\rho} \frac{\partial p}{\partial x_\alpha} \quad (10)$$

where p is the instantaneous pressure. Introducing (1) into the above and averaging over a wave period leads to

$$\begin{aligned} \frac{\partial V_{1\alpha}}{\partial t} + \frac{\partial}{\partial x_\beta} \left(V_{1\alpha} V_{1\beta} + \tilde{V}_\alpha \tilde{V}_\beta + V_{1\alpha} \tilde{V}_\beta + V_{1\beta} \tilde{V}_\alpha + \overline{u_{w\alpha} u_{w\beta}} + \overline{u'_\alpha u'_\beta} \right) \\ + \frac{\partial}{\partial z} \left(V_{1\alpha} W + \tilde{V}_\alpha W + \overline{u_{w\alpha} w} + \overline{u'_\alpha w'} \right) = \frac{-1}{\rho} \frac{\partial \bar{p}}{\partial x_\alpha} \end{aligned} \quad (11)$$

We assume that the the short-wave-averaged pressure is hydrostatic

$$\bar{p} = \rho g (\bar{\zeta} - z) \quad (12)$$

and that the Reynolds' stresses may be modeled using an eddy viscosity closure as follows

$$\overline{u'_\alpha u'_\beta} = -\nu_t \left(\frac{\partial V_\alpha}{\partial x_\beta} + \frac{\partial V_\beta}{\partial x_\alpha} \right) \quad (13)$$

$$\overline{u'_\alpha w'} = -\nu_t \left(\frac{\partial V_\alpha}{\partial z} + \frac{\partial W}{\partial x_\alpha} \right) \quad (14)$$

$$= -\nu_t \left(\frac{\partial V_{1\alpha}}{\partial z} + \frac{\partial W}{\partial x_\alpha} \right) \quad (15)$$

where ν_t is a time-independent eddy viscosity and V_α is the total short-wave-averaged velocity ($= \tilde{V}_\alpha + V_{1\alpha}$). Previous studies have shown that such a closure predicts the vertical structure of the nearshore currents reasonably accurately [*e.g.*, Svendsen & Hansen (1988) for cross-shore currents and Svendsen & Putrevu (1994) for longshore currents].

With these assumptions the equation governing $V_{1\alpha}$ reduces to

$$\begin{aligned} \frac{\partial V_{1\alpha}}{\partial t} - \frac{\partial}{\partial z} \left(\nu_t \frac{\partial V_{1\alpha}}{\partial z} \right) = & - \left(\frac{\partial \tilde{V}_\alpha}{\partial t} + \tilde{V}_\beta \frac{\partial \tilde{V}_\alpha}{\partial x_\beta} + g \frac{\partial \bar{\zeta}}{\partial x_\alpha} + f_\alpha \right) \\ & - \left(\tilde{V}_\beta \frac{\partial V_{1\alpha}}{\partial x_\beta} + V_{1\beta} \frac{\partial \tilde{V}_\alpha}{\partial x_\beta} + V_{1\beta} \frac{\partial V_{1\alpha}}{\partial x_\beta} + W \frac{\partial V_{1\alpha}}{\partial z} \right) \\ & + \frac{\partial}{\partial x_\beta} \left[\nu_t \left(\frac{\partial V_\alpha}{\partial x_\beta} + \frac{\partial V_\beta}{\partial x_\alpha} \right) \right] + \frac{\partial}{\partial z} \left(\nu_t \frac{\partial W}{\partial x_\alpha} \right) \end{aligned} \quad (16)$$

In the above, we have used the local continuity equation ($\partial V_\alpha / \partial x_\alpha + \partial W / \partial z = 0$) and defined f_α (which is the local contribution to the radiation stress) by

$$f_\alpha = \frac{\partial \overline{u_{w\beta} u_{w\alpha}}}{\partial x_\beta} + \frac{\partial \overline{w_w u_{w\alpha}}}{\partial z} \quad (17)$$

Using (4), the depth-integrated momentum equation (9) may be written as

$$\begin{aligned} \frac{\partial \tilde{V}_\alpha}{\partial t} + \tilde{V}_\beta \frac{\partial \tilde{V}_\alpha}{\partial x_\beta} + g \frac{\partial \bar{\zeta}}{\partial x_\alpha} = & \frac{-1}{\rho h} \frac{\partial S_{\alpha\beta}}{\partial x_\beta} - \frac{\tau_\alpha^B}{\rho h} - \frac{1}{\rho h} \frac{\partial T_{\alpha\beta}}{\partial x_\beta} \\ & - \frac{1}{h} \frac{\partial}{\partial x_\beta} \left[\int_{-h_0}^{\bar{\zeta}} V_{1\alpha} V_{1\beta} dz + V_{1\beta}(\bar{\zeta}) Q_{w\alpha} + V_{1\alpha}(\bar{\zeta}) Q_{w\beta} \right] \end{aligned} \quad (18)$$

so that (16) may be expressed as

$$\begin{aligned} \frac{\partial V_{1\alpha}}{\partial t} - \frac{\partial}{\partial z} \left(\nu_t \frac{\partial V_{1\alpha}}{\partial z} \right) = & \left\{ \frac{1}{\rho h} \frac{\partial S_{\alpha\beta}}{\partial x_\beta} - f_\alpha + \frac{\tau_\alpha^B}{\rho h} \right\} \\ & - \left\{ \left(\tilde{V}_\beta \frac{\partial V_{1\alpha}}{\partial x_\beta} + V_{1\beta} \frac{\partial \tilde{V}_\alpha}{\partial x_\beta} + V_{1\beta} \frac{\partial V_{1\alpha}}{\partial x_\beta} + W \frac{\partial V_{1\alpha}}{\partial z} \right) \right. \\ & \left. - \frac{1}{h} \frac{\partial}{\partial x_\beta} \left[\int_{-h_0}^{\bar{\zeta}} V_{1\alpha} V_{1\beta} dz + V_{1\beta}(\bar{\zeta}) Q_{w\alpha} + V_{1\alpha}(\bar{\zeta}) Q_{w\beta} \right] \right\} \\ & + \left\{ \frac{\partial}{\partial x_\beta} \left[\nu_t \left(\frac{\partial V_\alpha}{\partial x_\beta} + \frac{\partial V_\beta}{\partial x_\alpha} \right) \right] + \frac{\partial}{\partial z} \left(\nu_t \frac{\partial W}{\partial x_\alpha} \right) + \frac{1}{\rho h} \frac{\partial T_{\alpha\beta}}{\partial x_\beta} \right\} \end{aligned} \quad (19)$$

Equation 19 governs the vertical structure of $V_{1\alpha}$. Here we will solve (19) by applying an initial condition and the following conditions in the vertical

$$\nu_t \frac{\partial V_{1\alpha}}{\partial z} \Big|_{z=-h_0} = \frac{\tau_\alpha^B}{\rho}, \quad \int_{-h_0}^{\bar{\zeta}} V_{1\alpha} dz = -Q_{w\alpha} \quad (20)$$

This will lead to a solution for $V_{1\alpha}$ in which \tilde{V}_α will appear as an unknown parameter. We intend to use this solution in combination with the depth-integrated equations by substituting the result for $V_{1\alpha}$ into (9). Together with the continuity equation (4), this step will lead to a system of equations in which the only unknowns are $\bar{\zeta}$ and \tilde{V}_α . It is worth noticing that when we solve (19) by specifying the conditions (20), the resulting solution for $V_{1\alpha}$ will also automatically satisfy the correct condition for the shear stress at the surface ($\tau^S = 0$ in our case). This is a consequence of the fact that (19) represents the correct local momentum balance at any vertical location, and, at the same time, incorporates the information about the depth integrated total balance, which was introduced through the elimination of some of the terms in (19) using the depth-integrated equation (9).

To clearly expose the sizes of the various terms on the RHS of (19), we introduce the following non-dimensional variables

$$\begin{aligned} h &= h_b \ h^*, \quad z = h_b \ z^*, \quad x_\alpha = L \ x_\alpha^*, \quad t = T \ t^* \\ V_{1\alpha} &= \delta \ c_b \ V_{1\alpha}^*, \quad \bar{\zeta} = \delta \ h_b \ \bar{\zeta}^*, \quad S_{\alpha\beta} = \delta \ \rho c_b^2 h_b \ S_{\alpha\beta}^* \\ f_\alpha &= \delta \ \rho c_b^2 \ f_\alpha^*, \quad Q_{w\alpha} = \delta \ c_b h_b \ Q_{w\alpha}^*, \quad \tilde{V}_\alpha = \kappa \ c_b \ \tilde{V}_\alpha^* \\ W &= (\kappa + \delta) \frac{h_b}{L} c_b \ W^*, \quad \tau_\alpha^B = (\kappa + \delta) f_w \ \rho c_b^2 \ \tau_\alpha^{B*}, \quad \nu_t = \epsilon \ h_b c_b \ \nu_t^* \end{aligned} \quad (21)$$

where T is a typical time scale, h_b ($\sim 1\text{m}$) is a typical vertical length scale, L ($\sim 100\text{m}$) is a typical horizontal length scale, and $c_b = \sqrt{gh_b}$ is a typical wave celerity. In the above, all starred quantities are expected to be order 1. The nondimensional parameters ($\delta, \kappa, \epsilon, f_w$) that appear in some of the definitions represent the sizes of the physical quantities measured in terms of the chosen scales. δ (~ 0.1) represents the size of the short-wave-induced quantities; ϵ represents the size $\nu_t/h\sqrt{gh}$ in the nearshore region; f_w (~ 0.01) is the friction factor; and κ (with typical values between 0 and 0.3) represents the size of the current.

In terms of these dimensionless variables (19) may be written as

$$\frac{L}{c_b T} \frac{\partial V_{1\alpha}^*}{\partial t^*} - \frac{\epsilon L}{h_b} \frac{\partial}{\partial z^*} \left(\nu_t^* \frac{\partial V_{1\alpha}^*}{\partial z^*} \right) = I + II + III \quad (22)$$

where

$$I = \frac{1}{h^*} \frac{\partial S_{\alpha\beta}^*}{\partial x_\beta^*} - f_\alpha^* + \frac{f_w (\kappa + \delta) L}{\delta h_b} \tau_\alpha^{B*} \quad (23)$$

$$\begin{aligned} II &= -\kappa \left(\tilde{V}_\beta^* \frac{\partial V_{1\alpha}^*}{\partial x_\beta^*} + V_{1\beta}^* \frac{\partial \tilde{V}_\alpha^*}{\partial x_\beta^*} \right) - (\kappa + \delta) W^* \frac{\partial V_{1\alpha}^*}{\partial z^*} - \delta V_{1\beta}^* \frac{\partial V_{1\alpha}^*}{\partial x_\beta^*} \\ &\quad + \delta \frac{1}{h^*} \frac{\partial}{\partial x_\beta^*} \left[\int_{-h_0^*}^{\delta \bar{\zeta}^*} V_{1\alpha}^* V_{1\beta}^* dz^* + V_{1\beta}^* (\bar{\zeta}^*) Q_{w\alpha}^* + V_{1\alpha}^* (\bar{\zeta}^*) Q_{w\beta}^* \right] \end{aligned} \quad (24)$$

$$III = \frac{\epsilon (\kappa + \delta) h_b}{L} \left\{ \frac{\partial}{\partial x_\beta^*} \left[\nu_t^* \left(\frac{\partial V_\alpha^*}{\partial x_\beta^*} + \frac{\partial V_\beta^*}{\partial x_\alpha^*} \right) \right] + \frac{\partial}{\partial z^*} \left[\nu_t^* \left(\frac{\partial W^*}{\partial x_\alpha^*} \right) \right] \right\}$$

$$-\frac{\partial}{\partial x_\beta^*} \left[\int_{-h_0^*}^{\overline{\delta\zeta^*}} \nu_t^* \left(\frac{\partial V_\alpha^*}{\partial x_\beta^*} + \frac{\partial V_\beta^*}{\partial x_\alpha^*} \right) dz^* \right] \} \quad (25)$$

The definitions of the scales imply that $\epsilon L/h_b \sim 1$. For both longshore currents and undertow on an alongshore uniform beach, the second term on the LHS of (22) is of the same order of magnitude as the combination of terms in I . It is therefore consistent to assume that together the terms in I and the second term on the LHS of (22) are of the same order of magnitude. $L/c_b T$ is an independent parameter – it is zero for steady flows and will be an order one quantity for the fastest of the infragravity motions. For generality, this parameter is assumed to be order 1 here.

For $\kappa, \delta \ll 1$, the terms represented by II on the RHS of (22) are an order of magnitude smaller than the terms represented by I . Finally, the terms represented by III are much smaller than those represented by I and II . To keep the solution as general as possible, we assume from here on that $\kappa \sim \delta$. (The cases $\kappa \ll \delta$ and $\kappa \gg \delta$ form subsets of the solution given below.)

Equation 22 suggests that $V_{1\alpha}^*$ may be solved using a perturbation expansion of the type

$$V_{1\alpha}^* = V_{1\alpha}^{*(0)} + \delta V_{1\alpha}^{*(1)} + \dots \quad (26)$$

with $V_{1\alpha}^{*(0)}$ governed by

$$\frac{L}{c_b T} \frac{\partial V_{1\alpha}^{*(0)}}{\partial t} - \frac{\epsilon L}{h_b} \frac{\partial}{\partial z^*} \left(\nu_t^* \frac{\partial V_{1\alpha}^{*(0)}}{\partial z^*} \right) = \frac{1}{h^*} \frac{\partial S_{\alpha\beta}^*}{\partial x_\beta^*} - f_\alpha^* + \frac{f_w(\kappa + \delta)L}{\delta h_b} \tau_\alpha^{B*} \quad (27)$$

The conditions associated with (27) are

$$\left(\nu_t^* \frac{\partial V_{1\alpha}^{*(0)}}{\partial z^*} \right)_{z^* = -h_0^*} = \frac{f_w(\kappa + \delta)}{\epsilon \delta} \tau_\alpha^{B*}, \quad \int_{-h_0^*}^{\overline{\delta\zeta^*}} V_{1\alpha}^{*(0)} dz = -Q_{w\alpha}^* \quad (28)$$

and an appropriate initial condition.

In the steady case the $V_{1\alpha}^{(0)}$ problem described above is completely analogous to the familiar undertow problem (Svendsen 1984): $V_{1\alpha}^{(0)}$ is forced by the local imbalance between the depth-averaged and local values of the forcing and, integrated over depth, $V_{1\alpha}^{(0)}$ compensates for the short-wave-induced volume flux. Thus, $V_{1\alpha}^{(0)}$ may be interpreted as a generalized, time-varying undertow.

The $V_{1\alpha}^{*(1)}$ problem reduces to the following

$$\begin{aligned} & \frac{L}{c_b T} \frac{\partial V_{1\alpha}^{*(1)}}{\partial t} - \frac{\epsilon L}{h_b} \frac{\partial}{\partial z^*} \left(\nu_t^* \frac{\partial V_{1\alpha}^{*(1)}}{\partial z^*} \right) = \\ & - \left(\tilde{V}_\beta^* \frac{\partial V_{1\alpha}^{*(0)}}{\partial x_\beta^*} + V_{1\beta}^{*(0)} \frac{\partial \tilde{V}_\alpha^*}{\partial x_\beta^*} \right) - W^* \frac{\partial V_{1\alpha}^{*(0)}}{\partial z^*} - V_{1\beta}^{*(0)} \frac{\partial V_{1\alpha}^{*(0)}}{\partial x_\beta^*} \\ & - \frac{1}{h^*} \frac{\partial}{\partial x_\beta^*} \left[\int_{-h_0^*}^{\overline{\delta\zeta^*}} V_{1\alpha}^{*(0)} V_{1\beta}^{*(0)} dz^* + V_{1\beta}^{*(0)} (\overline{\delta\zeta^*}) Q_{w\alpha}^* + V_{1\alpha}^{*(0)} (\overline{\delta\zeta^*}) Q_{w\beta}^* \right] \end{aligned} \quad (29)$$

subject to the conditions

$$\left. \nu_t^* \frac{\partial V_{1\alpha}^{*(1)}}{\partial z^*} \right)_{z=-h_0^*} = 0, \quad \int_{-h_0}^{\bar{\delta}\zeta^*} V_{1\alpha}^{*(1)} dz^* = 0 \quad (30)$$

and an initial condition.

Below we discuss the solutions for $V_{1\alpha}^{(0)}$ and $V_{1\alpha}^{(1)}$.

3.1 Solution for $V_{1\alpha}^{(0)}$

Here we outline the solution for $V_{1\alpha}^{(0)}$. Returning to dimensional variables, the equation governing $V_{1\alpha}^{(0)}$ is

$$\frac{\partial V_{1\alpha}^{(0)}}{\partial t} - \frac{\partial}{\partial z} \left(\nu_t \frac{\partial V_{1\alpha}^{(0)}}{\partial z} \right) = F_\alpha \quad (31)$$

where F_α represents the difference between the depth-averaged and local values of the forcing on a fluid element and is given by

$$F_\alpha = \frac{1}{\rho h} \frac{\partial S_{\alpha\beta}}{\partial x_\beta} - f_\alpha + \frac{\tau_\alpha^B}{\rho h} \quad (32)$$

The conditions associated with (31) are

$$\left. \nu_t \frac{\partial V_{1\alpha}^{(0)}}{\partial z} \right)_{z=-h_0} = \frac{\tau_\alpha^B}{\rho}, \quad \int_{-h_0}^{\bar{\zeta}} V_{1\alpha}^{(0)} dz = -Q_{w\alpha} \quad (33)$$

and an appropriate initial condition.

The solution for $V_{1\alpha}^{(0)}$ can be derived by dividing $V_{1\alpha}^{(0)}$ into a part that satisfies the inhomogeneous conditions (33) and an additional contribution $V_{1\alpha}^{(0,1)}$. Hence,

$$V_{1\alpha}^{(0)} = \frac{-Q_{w\alpha}}{h} + \frac{\tau_\alpha^B}{\rho} \left(\int_{-h_0}^z \frac{dz'}{\nu_t} - \frac{1}{h} \int_{-h_0}^{\bar{\zeta}} \int_{-h_0}^z \frac{dz'}{\nu_t} dz \right) + V_{1\alpha}^{(0,1)} \quad (34)$$

Substitution of (34) into (31) leads to the following equation for $V_{1\alpha}^{(0,1)}$

$$\frac{\partial V_{1\alpha}^{(0,1)}}{\partial t} - \frac{\partial}{\partial z} \left(\nu_t \frac{\partial V_{1\alpha}^{(0,1)}}{\partial z} \right) = F_\alpha - \frac{\partial}{\partial t} \left(\frac{Q_{w\alpha}}{h} \right) + \frac{1}{\rho} \frac{\partial}{\partial t} \left[\tau_\alpha^B \left(\int_{-h_0}^z \frac{dz'}{\nu_t} - \frac{1}{h} \int_{-h_0}^{\bar{\zeta}} \int_{-h_0}^z \frac{dz'}{\nu_t} dz \right) \right] \quad (35)$$

The conditions on $V_{1\alpha}^{(0,1)}$ are

$$\left. \nu_t \frac{\partial V_{1\alpha}^{(0,1)}}{\partial z} \right)_{z=-h_0} = 0, \quad \int_{-h_0}^{\bar{\zeta}} V_{1\alpha}^{(0,1)} dz = 0 \quad (36)$$

along with an appropriate initial condition.

The solution for $V_{1\alpha}^{(0,1)}$ is given by (see Appendix A)

$$\begin{aligned} V_{1\alpha}^{(0,1)} &= V_{s\alpha}^{(0)}(z, t) \\ &+ \left[\int_{-h_0}^z \frac{1}{\nu_t} \int_{-h_0}^{z_1} \frac{\partial V_{s\alpha}^{(0)}(z_2, t)}{\partial t} dz_2 dz_1 - \frac{1}{h} \int_{-h_0}^{\bar{\zeta}} \int_{-h_0}^z \frac{1}{\nu_t} \int_{-h_0}^{z_1} \frac{\partial V_{s\alpha}^{(0)}(z_2, t)}{\partial t} dz_2 dz_1 dz \right] \end{aligned}$$

$$\begin{aligned}
& + \left[\int_{-h_0}^z \frac{1}{\nu_t} \int_{-h_0}^{z_1} \int_{-h_0}^{z_2} \frac{1}{\nu_t} \int_{-h_0}^{z_3} \frac{\partial^2 V_{s\alpha}^{(0)}(z_4, t)}{\partial t^2} dz_4 dz_3 dz_2 dz_1 \right. \\
& \left. - \frac{1}{h} \int_{-h_0}^{\bar{\zeta}} \int_{-h_0}^z \frac{1}{\nu_t} \int_{-h_0}^{z_1} \int_{-h_0}^{z_2} \frac{1}{\nu_t} \int_{-h_0}^{z_3} \frac{\partial^2 V_{s\alpha}^{(0)}(z_4, t)}{\partial t^2} dz_4 dz_3 dz_2 dz_1 dz \right] \\
& + \dots
\end{aligned} \tag{37}$$

where

$$V_{s\alpha}^{(0)}(z, t) = - \int_{-h_0}^z \frac{1}{\nu_t} \int_{-h_0}^{z_1} R_{\alpha}^{(0)}(z_2, t) dz_2 dz_1 + \frac{1}{h} \int_{-h_0}^{\bar{\zeta}} \int_{-h_0}^z \frac{1}{\nu_t} \int_{-h_0}^{z_1} R_{\alpha}^{(0)}(z_2, t) dz_2 dz_1 dz \tag{38}$$

$$R_{\alpha}^{(0)} = F_{\alpha} - \frac{\partial}{\partial t} \left(\frac{Q_{w\alpha}}{h} \right) + \frac{1}{\rho} \frac{\partial}{\partial t} \left[\tau_{\alpha}^B \left(\int_{-h_0}^z \frac{dz'}{\nu_t} - \frac{1}{h} \int_{-h_0}^{\bar{\zeta}} \int_{-h_0}^z \frac{dz'}{\nu_t} dz \right) \right] \tag{39}$$

Note that $V_{s\alpha}^{(0)}(z, t)$ is the quasi-steady solution for $V_{1\alpha}^{(0,1)}$ (it represents the the steady state response to the instantaneous value of the forcing). Also note that each successive term in the expansion (37) is of magnitude $h_b^2/\nu_0 T$ times the previous term (where ν_0 is the magnitude of the eddy viscosity). Thus, if

$$h_b^2/\nu_0 T < 1 \tag{40}$$

(a condition that is satisfied for short-wave-averaged motions in the nearshore, *e.g.*, infra-gravity waves), then $V_{s\alpha}^{(0)}(z, t)$, the quasi-steady solution given by (38), represents the first approximation to the complete solution.

Finally the solution for $V_{1\alpha}^{(0)}$ is obtained by substituting (37) into (34). In the following, we assume that the solutions for $V_{1\alpha}^{(0,1)}$ and $V_{1\alpha}^{(0)}$ are known.

3.2 Solution for $V_{1\alpha}^{(1)}$

The problem for $V_{1\alpha}^{(1)}$ is identical to the $V_{1\alpha}^{(0,1)}$ problem, and a solution can be obtained by the method outlined in Appendix A. Defining,

$$\begin{aligned}
R_{\alpha}^{(1)}(z, t) = & - \left(\tilde{V}_{\beta} \frac{\partial V_{1\alpha}^{(0)}}{\partial x_{\beta}} + V_{1\beta}^{(0)} \frac{\partial \tilde{V}_{\alpha}}{\partial x_{\beta}} + W \frac{\partial V_{1\alpha}^{(0)}}{\partial z} + V_{1\beta}^{(0)} \frac{\partial V_{1\alpha}^{(0)}}{\partial x_{\beta}} \right) \\
& + \frac{1}{h} \frac{\partial}{\partial x_{\beta}} \left[\int_0^h V_{1\alpha}^{(0)} V_{1\beta}^{(0)} dz + V_{1\beta}^{(0)}(\bar{\zeta}) Q_{w\alpha} + V_{1\alpha}^{(0)}(\bar{\zeta}) Q_{w\beta} \right]
\end{aligned} \tag{41}$$

we can write the solution for $V_{1\alpha}^{(1)}$ as

$$V_{1\alpha}^{(1)} = - \int_{-h_0}^z \frac{1}{\nu_t} \int_{-h_0}^{z_1} R_{\alpha}^{(1)}(z_2, t) dz_2 dz_1 + \frac{1}{h} \int_{-h_0}^{\bar{\zeta}} \int_{-h_0}^z \frac{1}{\nu_t} \int_{-h_0}^{z_1} R_{\alpha}^{(1)}(z_2, t) dz_2 dz_1 dz + H.O.T. \tag{42}$$

where *H.O.T.* represents the higher order terms given by the integrals similar to those in (37).

Equation 42, with the higher order terms neglected, will be used in the calculations below.

Thus, the results below represent the first approximation to the complete result.

4 Results for the integrals

The integrals required in (9) can now be calculated as follows

$$\begin{aligned} \int_{-h_0}^{\bar{\zeta}} V_{1\alpha} V_{1\beta} dz + V_{1\beta}(\bar{\zeta}) Q_{w\alpha} + V_{1\alpha}(\bar{\zeta}) Q_{w\beta} &= \int_{-h_0}^{\bar{\zeta}} V_{1\alpha}^{(0)} V_{1\beta}^{(0)} dz + V_{1\beta}^{(0)}(\bar{\zeta}) Q_{w\alpha} + V_{1\alpha}^{(0)}(\bar{\zeta}) Q_{w\beta} \\ &+ \int_{-h_0}^{\bar{\zeta}} \left(V_{1\alpha}^{(0)} V_{1\beta}^{(1)} + V_{1\alpha}^{(1)} V_{1\beta}^{(0)} \right) dz \\ &+ V_{1\beta}^{(1)}(\bar{\zeta}) Q_{w\alpha} + V_{1\alpha}^{(1)}(\bar{\zeta}) Q_{w\beta} + O(V_1^{(1)})^2 \end{aligned} \quad (43)$$

Substitution of the results for $V_{1\alpha}^{(1)}$ leads to the following result as the first approximation for the integrals (Appendix B)

$$\begin{aligned} \int_{-h_0}^{\bar{\zeta}} V_{1\alpha} V_{1\beta} dz + V_{1\beta}(\bar{\zeta}) Q_{w\alpha} + V_{1\alpha}(\bar{\zeta}) Q_{w\beta} &= M_{\alpha\beta} + A_{\alpha\beta\delta} \tilde{V}_\delta \\ &- h \left(D_{\delta\beta} \frac{\partial \tilde{V}_\alpha}{\partial x_\delta} + D_{\delta\alpha} \frac{\partial \tilde{V}_\beta}{\partial x_\delta} + B_{\alpha\beta} \frac{\partial \tilde{V}_\delta}{\partial x_\delta} \right) \end{aligned} \quad (44)$$

where the tensors A , B , D , and M are defined in Appendix B (equations 96-99).

Substituting (44) into (9) we get

$$\begin{aligned} \frac{\partial}{\partial t} (\tilde{V}_\alpha h) + \frac{\partial}{\partial x_\beta} (\tilde{V}_\alpha \tilde{V}_\beta h + A_{\alpha\beta\delta} \tilde{V}_\delta) + \frac{1}{\rho} \frac{\partial}{\partial x_\beta} (S_{\alpha\beta} + \rho M_{\alpha\beta}) + gh \frac{\partial \bar{\zeta}}{\partial x_\alpha} + \frac{\tau_\alpha^B}{\rho} \\ + \frac{\partial}{\partial x_\beta} \left[T_{\alpha\beta} - h \left(D_{\delta\beta} \frac{\partial \tilde{V}_\alpha}{\partial x_\delta} + D_{\delta\alpha} \frac{\partial \tilde{V}_\beta}{\partial x_\delta} \right) \right] - \frac{\partial}{\partial x_\beta} \left(h B_{\alpha\beta} \frac{\partial \tilde{V}_\delta}{\partial x_\delta} \right) = 0 \end{aligned} \quad (45)$$

Thus, as in SP94, we find that the vertical variation of the short-wave-averaged horizontal velocities leads to mixing-like terms in the depth-integrated momentum equation. An attractive feature of this result is that the resulting dispersion tensor can be readily calculated. This is important because, as in SP94, the lateral mixing caused by the shear dispersion mechanism is expected to dominate the lateral mixing in the nearshore even in the presence of turbulence from breaking waves.

Equations 6.1a-c of Smith (1997) are analogous to our (44). There are a few differences between these two sets of equations. The most important difference between our equations and Smith's is that in our case the vertical integral of $V_{1\alpha}^{(0)}$ is non-zero where it is zero in Smith's case. A second difference is that we have a few extra terms (the $B_{\alpha\beta}$ and $A_{\alpha\beta\delta}$ terms) that Smith does not have. Smith does not get these terms because he essentially neglects the terms $\tilde{V}_\beta \partial V_{1\alpha} / \partial x_\beta$ and $W \partial V_{1\alpha} / \partial z$ in (29) and assumes that $\partial \tilde{V}_\beta / \partial x_\beta = 0$. As shown in the next section, the consequences of these simplifications are likely to be minor.

Before proceeding further, we note that the expression for the dispersion tensor, $D_{\alpha\beta}$, can be rewritten in a form that is similar to the one given in Fischer *et al.* (1979) for the dispersion of solutes. To do that, we define

$$p = \int_{-h_0}^z V_{1\alpha}^{(0)} dz' \quad \text{and} \quad q = \int_z^{\bar{\zeta}} \frac{1}{v_t} \int_{-h_0}^{z'} V_{1\beta}^{(0)} dz'' dz' \quad (46)$$

so that

$$\frac{1}{h} \int_{-h_0}^{\bar{\zeta}} \frac{1}{\nu_t} \left(\int_{-h_0}^z V_{1\alpha}^{(0)} dz' \right) \left(\int_{-h_0}^z V_{1\beta}^{(0)} dz'' \right) dz = - \int_{-h_0}^{\bar{\zeta}} p dq \quad (47)$$

$$= - p q \Big|_{-h_0}^{\bar{\zeta}} + \int_{-h_0}^{\bar{\zeta}} q dp \quad (48)$$

$$= \frac{1}{h} \int_{-h_0}^{\bar{\zeta}} V_{1\alpha}^{(0)} \int_z^{\bar{\zeta}} \frac{1}{\nu_t} \int_{-h_0}^{z'} V_{1\beta}^{(0)} dz'' dz' dz \quad (49)$$

The boundary terms vanish since $p(-h_0) = q(\bar{\zeta}) = 0$. Therefore,

$$D_{\alpha\beta} = \frac{1}{h} \int_{-h_0}^{\bar{\zeta}} V_{1\alpha}^{(0)} \int_z^{\bar{\zeta}} \frac{1}{\nu_t} \int_{-h_0}^{z'} V_{1\beta}^{(0)} dz'' dz' dz \quad (50)$$

The structure of the expression (50) for the dispersion tensor is similar to that given in Fischer *et al.* (1979, eq. 4.64) for the dispersion of solutes in a two-dimensional shear flow, showing the close analogy between the shear-dispersion of momentum considered here and the shear-dispersion of solutes initially considered by Taylor and expanded on by others [see Fischer *et al.* (1979, Chapter 4) for a discussion of the solute dispersion problem].

5 Discussion

The results derived above show that the vertical nonuniformity of the short-wave-averaged velocities leads to mixing-like terms in the depth-integrated momentum equation. The mechanism by which this happens (the combination of vertical mixing and horizontal advection) is identical to Taylor's (1953, 1954) shear dispersion mechanism for solutes; the results summarized by Fischer *et al.* are generalizations of Taylor's results to two horizontal dimensions.

Equation (45) shows that the $M_{\alpha\beta}$ term modifies the radiation stress term², the $A_{\alpha\beta\delta}$ term modifies the convective acceleration term, and the $D_{\alpha\beta}$ and $B_{\alpha\beta}$ terms modify the lateral mixing term. Below, we estimate the importance of these modifications by first estimating the size of the various tensors and then comparing these sizes to the magnitude of the terms they modify. Based on the second of the conditions (28) we use the estimate $V_{1\alpha}^{(0)} \sim -Q_{w\alpha}/h$ [see also (34)]. In the calculations, we also use the following estimates (which are reasonable for typical nearshore conditions): $\nu_t \sim 0.01h\sqrt{gh}$, $Q_w \sim 0.1H^2\sqrt{gh}/h$, $H \sim 0.6h$, $h \sim 1$ m, $L \sim 100$ m. The various tensors have the following magnitudes

$$M_{\alpha\beta} \sim V_1^2 h \sim \frac{Q_w^2}{h} \quad (51)$$

²Probably a more natural interpretation of the $M_{\alpha\beta}$ term is that it is analogous to the momentum correction factor of hydraulics. The interpretation chosen here essentially follows from Phillips (1977). In fact, Phillips defines the radiation stress (eq. 3.6.12, p. 62) by including a term $(\rho Q_{w\alpha} Q_{w\beta}/h)$ that represents the first approximation to $M_{\alpha\beta}$.

$$A_{\alpha\beta\delta} \sim \frac{Q_w^2 h}{\nu_t L} \quad (52)$$

$$B_{\alpha\beta} \sim D_{\alpha\beta} \sim \frac{Q_w^2}{\nu_t} \quad (53)$$

As mentioned above, the $M_{\alpha\beta}$ term may be thought of as modifying the radiation stress term. Thus, it is reasonable to estimate the importance of the $M_{\alpha\beta}$ term by comparing its size with the typical size of $S_{\alpha\beta}$. Using the estimates mentioned earlier, we have

$$M_{\alpha\beta} \sim 0.01 \left(\frac{H}{h} \right)^4 gh^2 \ll S_{\alpha\beta} \sim 0.1gh^2 \quad (54)$$

Thus, we expect that the effect of the $M_{\alpha\beta}$ term will be small.

Next, let us consider the $A_{\alpha\beta\delta}$ term. This term modifies the convective acceleration term and has a magnitude

$$A_{\alpha\beta\delta} \tilde{V}_\delta \sim \frac{Q_w^2 h \bar{V}}{\nu_t L} \quad (55)$$

where \bar{V} is a typical value of the current. In comparison, the convective acceleration term has the following magnitude

$$\tilde{V}_\alpha \tilde{V}_\beta h \sim h \bar{V}^2 \quad (56)$$

Therefore, as long as

$$\frac{\bar{V}}{\sqrt{gh}} \gg \left(\frac{H}{h} \right)^4 \frac{h}{L} \sim 10^{-3} \quad (57)$$

the $A_{\alpha\beta\delta}$ will not modify the convective acceleration term significantly. Since, nearshore currents typically have sizes that satisfy (57), we expect that the modification of the convective acceleration term due to the $A_{\alpha\beta\delta}$ term will be minor.

To discuss the effects of the $B_{\alpha\beta}$ term, it is convenient to first rewrite this term using the continuity equation (4) as

$$-\frac{\partial}{\partial x_\beta} \left(h B_{\alpha\beta} \frac{\partial \tilde{V}_\delta}{\partial x_\delta} \right) = \frac{\partial}{\partial x_\beta} \left(B_{\alpha\beta} \frac{\partial \bar{\zeta}}{\partial t} + B_{\alpha\beta} V_\delta \frac{\partial h}{\partial x_\delta} \right) \quad (58)$$

Thus, the $B_{\alpha\beta}$ term modifies both the temporal and convective acceleration terms. Therefore, we have to compare $\partial/\partial x_\beta (B_{\alpha\beta} \partial \bar{\zeta}/\partial t)$ term with the $V_\alpha \partial \bar{\zeta}/\partial t$ and $\partial/\partial x_\beta (B_{\alpha\beta} V_\delta \partial h/\partial x_\delta)$ with the $\partial/\partial x_\beta (\tilde{V}_\alpha \tilde{V}_\beta h)$ term to estimate the importance of the $B_{\alpha\beta}$ term. We have

$$\frac{\partial}{\partial x_\beta} \left(B_{\alpha\beta} \frac{\partial \bar{\zeta}}{\partial t} \right) \sim \left(\frac{H}{h} \right)^4 \frac{h \sqrt{gh}}{L} \frac{\bar{\zeta}}{T} \quad (59)$$

whereas

$$V_\alpha \frac{\partial \bar{\zeta}}{\partial t} \sim \bar{V} \frac{\bar{\zeta}}{T} \quad (60)$$

Comparing (59) and (60), we deduce that the $V_\alpha \partial \bar{\zeta}/\partial t$ term will dominate over the $B_{\alpha\beta} \partial \bar{\zeta}/\partial t$ term as long as (57) is satisfied. A comparison of $\partial/\partial x_\beta (B_{\alpha\beta} V_\delta \partial h/\partial x_\delta)$ with the convective acceleration terms leads to a similar result.

Therefore, we expect that the modifications caused by the $A_{\alpha\beta\delta}$, $M_{\alpha\beta}$, and $B_{\alpha\beta}$ terms will be small under typical nearshore conditions. In contrast, we expect that the effect of the $D_{\alpha\beta}$ terms will be significant. The $D_{\alpha\beta}$ terms modify the turbulent lateral mixing. Thus, we have to compare the size of $D_{\alpha\beta}$ with ν_t to estimate the importance of this additional lateral mixing. We have

$$D_{\alpha\beta} \sim \frac{Q_{w\alpha}Q_{w\beta}}{\nu_t} \sim \left(\frac{H}{h}\right)^4 h\sqrt{gh} \gg \nu_t \quad (61)$$

Therefore, the lateral mixing due to the $D_{\alpha\beta}$ terms will dominate the total lateral mixing in the nearshore region.

Based on the analysis above, we conclude that the primary effect of the vertical nonuniformity of the short-wave-averaged horizontal velocities is an enhanced level of lateral mixing in the depth-integrated momentum equation, and that this enhanced mixing is primarily provided by the $D_{\alpha\beta}$ terms.

6 Concluding Remarks

In this paper we extended the results of SP94 to the general case of unsteady flow over an arbitrary bottom topography. The results show that the the vertical nonuniformity of the short-wave-averaged horizontal velocities leads to mixing-like terms for the horizontal velocity in the depth-integrated equations of momentum. An order of magnitude analysis shows that the lateral mixing effect caused by the vertical nonuniformity will dominate over the turbulent mixing.

All the discussions in this paper have dealt with circulations induced by short-waves in the nearshore region. However, the equations derived here for shear dispersion of momentum, as well as the primary conclusion that the vertical nonuniformity of the horizontal velocities leads to enhanced horizontal mixing, will hold for a number of other flows modeled by depth-integrated equations. In that context, it is interesting to notice that Blumberg & Mellor (1987) observed a similar effect for mesoscale phenomena in the coastal ocean in their numerical experiments using a three-dimensional model. Describing their results, they write, “The relatively fine vertical resolution used in the applications resulted in a reduced need for horizontal diffusion because horizontal advection followed by vertical mixing effectively acts like horizontal diffusion in a real physical sense.” Our interpretation of this statement is that by using a fine vertical resolution in their fully three-dimensional model Blumberg & Mellor represented the vertical variations of the horizontal velocities sufficiently accurately. Hence, the nonlinear terms in their equations automatically provided the lateral mixing calculated in this paper.

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A Derivation of the solution for $V_{1\alpha}^{(0,1)}$

The solution for $V_{1\alpha}^{(0,1)}$ for the case in which the eddy viscosity does not vary with the vertical coordinate can be found in the following way. For convenience, we define

$$R_{\alpha}^{(0)}(\xi, t) = F_{\alpha} - \frac{\partial}{\partial t} \left(\frac{Q_{w\alpha}}{h} \right) + \frac{1}{\rho} \frac{\partial}{\partial t} \left[\tau_{\alpha}^B \left(\int_{-h_0}^z \frac{dz'}{\nu_t} - \frac{1}{h} \int_{-h_0}^{\bar{\zeta}} \int_{-h_0}^z \frac{dz'}{\nu_t} dz \right) \right] \quad (62)$$

where $\xi = z + h_0$ and consider the problem of solving

$$\frac{\partial V_{1\alpha}^{(0,1)}}{\partial t} - \nu_t \frac{\partial^2 V_{1\alpha}^{(0,1)}}{\partial \xi^2} = R_{\alpha}^{(0)}(\xi, t) \quad (63)$$

subject to

$$\left. \frac{\partial V_{1\alpha}^{(0,1)}}{\partial \xi} \right|_{\xi=0} = 0, \quad \int_0^h V_{1\alpha}^{(0,1)} d\xi = 0, \quad V_{1\alpha}^{(0,1)}(\xi, t=0) = 0 \quad (64)$$

where a zero initial condition has been assumed for definiteness. The solution of the above is (Carslaw & Jaeger 1959, eq. 20, p. 32)

$$V_{1\alpha}^{(0,1)}(\xi, t) = \int_0^t \frac{\partial}{\partial t} [V^*(\xi, T, t-T)] dT \quad (65)$$

where V^* is the solution to the version of (63) with “steady” forcing given by

$$\frac{\partial V^*}{\partial t} - \nu_t \frac{\partial^2 V^*}{\partial \xi^2} = R_{\alpha}^{(0)}(\xi, T) \quad (66)$$

subject to

$$\left. \frac{\partial V^*}{\partial \xi} \right|_{\xi=0} = 0, \quad \int_0^h V^* d\xi = 0, \quad V^*(\xi, T, t=0) = 0 \quad (67)$$

The solution for V^* may be expressed as follows:

$$V^*(\xi, T, t) = V_s^*(\xi, T) + V_t^*(\xi, T, t) \quad (68)$$

where the subscripts s and t represent steady and transient parts, respectively. V_s^* is given by

$$V_s^*(\xi, T) = \frac{-1}{\nu_t} \int_0^{\xi} \int_0^{\xi_1} R_{\alpha}^{(0)}(\xi_2, T) d\xi_2 d\xi_1 + \frac{1}{h\nu_t} \int_0^h \int_0^{\xi} \int_0^{\xi_1} R_{\alpha}^{(0)}(\xi_2, T) d\xi_2 d\xi_1 d\xi \quad (69)$$

Note that V_s^* is a quasi-steady solution which represents the steady state response to the instantaneous value of the forcing $R_{\alpha}^{(0)}(\xi, t)$.

$V_t^*(\xi, T, t)$ is the solution to the homogeneous problem

$$\frac{\partial V_t^*}{\partial t} - \nu_t \frac{\partial^2 V_t^*}{\partial \xi^2} = 0 \quad (70)$$

satisfying the conditions

$$\left. \frac{\partial V_t^*}{\partial \xi} \right|_{\xi=0} = 0, \quad \int_0^h V_t^* d\xi = 0, \quad V_t^*(\xi, T, t=0) = -V_s^*(\xi, T) \quad (71)$$

The solution for the problem for $V_t^*(\xi, T, t)$ described by (70) subject to the conditions (71) can be written as

$$V_t^*(\xi, T, t) = \sum_{n=1}^{\infty} A_{\alpha}^{(n)}(T) \cos\left(\frac{n\pi\xi}{h}\right) \exp(-\lambda_n t) \quad (72)$$

where $\lambda_n = n^2\pi^2\nu_t/h^2$ and $A_{\alpha}^{(n)}(T)$ is given by

$$A_{\alpha}^{(n)}(T) = \frac{-2}{h} \int_0^h \cos\left(\frac{n\pi\xi}{h}\right) V_s^*(\xi, T) d\xi \quad (73)$$

Substitution of (69) and (72) into (65) leads to the following solution for $V_{1\alpha}^{(0,1)}$

$$V_{1\alpha}^{(0,1)} = - \sum_{n=1}^{\infty} \lambda_n \cos\left(\frac{n\pi\xi}{h}\right) \int_0^t A_{\alpha}^{(n)}(T) \exp[-\lambda_n(t-T)] dT \quad (74)$$

(74) is the exact solution for $V_{1\alpha}^{(0,1)}$. However, this form of the solution is not very transparent. The following manipulations are aimed at expressing (74) in a more transparent form.

Integrating (74) by parts and neglecting a term proportional to $\exp(-\lambda_n t)$, on the grounds that it decays after a sufficiently long time, leads to

$$V_{1\alpha}^{(0,1)} = - \sum_{n=1}^{\infty} \cos\left(\frac{n\pi\xi}{h}\right) A_{\alpha}^{(n)}(t) + \sum_{n=1}^{\infty} \cos\left(\frac{n\pi\xi}{h}\right) \int_0^t \frac{dA_{\alpha}^{(n)}}{dT} \exp[-\lambda_n(t-T)] dT \quad (75)$$

Substituting for $A_{\alpha}^{(n)}(t)$ in the first summation on the RHS of (75) we get

$$- \sum_{n=1}^{\infty} \cos\left(\frac{n\pi\xi}{h}\right) A_{\alpha}^{(n)}(t) = \frac{2}{h} \sum_{n=1}^{\infty} \cos\left(\frac{n\pi\xi}{h}\right) \int_0^h \cos\left(\frac{n\pi\xi_1}{h}\right) V_s^*(\xi_1, t) d\xi_1 \quad (76)$$

$$= V_s^*(\xi, t) \quad (77)$$

so that

$$V_{1\alpha}^{(0,1)} = V_s^*(\xi, t) + \sum_{n=1}^{\infty} \cos\left(\frac{n\pi\xi}{h}\right) \int_0^t \frac{dA_{\alpha}^{(n)}}{dT} \exp[-\lambda_n(t-T)] dT \quad (78)$$

The second term on the RHS in the equation above can be transformed as follows. Integrating the second term by parts and again neglecting a term proportional to $\exp(-\lambda_n t)$ we get

$$\begin{aligned} \sum_{n=1}^{\infty} \cos\left(\frac{n\pi\xi}{h}\right) \int_0^t \frac{dA_{\alpha}^{(n)}}{dT} \exp[-\lambda_n(t-T)] dT &= \\ \sum_{n=1}^{\infty} \cos\left(\frac{n\pi\xi}{h}\right) \frac{h^2}{n^2\pi^2\nu_t} \frac{dA_{\alpha}^{(n)}}{dt} &= \\ - \sum_{n=1}^{\infty} \cos\left(\frac{n\pi\xi}{h}\right) \frac{h^2}{n^2\pi^2\nu_t} \int_0^t \frac{d^2 A_{\alpha}^{(n)}}{dT^2} \exp[-\lambda_n(t-T)] dT &= \end{aligned} \quad (79)$$

We can express the first summation on the RHS of (79) in terms of $V_s^*(\xi, t)$ as follows. Let

$$G_1(\xi, t) = \sum_{n=1}^{\infty} \cos\left(\frac{n\pi\xi}{h}\right) \frac{h^2}{n^2\pi^2\nu_t} \frac{dA_\alpha^{(n)}}{dt} \quad (80)$$

Using (73) we then have

$$G_1(\xi, t) = \frac{-2}{h\nu_t} \sum_{n=1}^{\infty} \cos\left(\frac{n\pi\xi}{h}\right) \frac{h^2}{n^2\pi^2} \int_0^h \cos\left(\frac{n\pi\xi_1}{h}\right) \frac{\partial V_s^*(\xi_1, t)}{\partial t} d\xi_1 \quad (81)$$

Differentiating the above twice leads to

$$\frac{\partial^2 G_1}{\partial \xi^2} = \frac{2}{h\nu_t} \sum_{n=1}^{\infty} \cos\left(\frac{n\pi\xi}{h}\right) \int_0^h \cos\left(\frac{n\pi\xi_1}{h}\right) \frac{\partial V_s^*(\xi_1, t)}{\partial t} d\xi_1 \quad (82)$$

$$= \frac{1}{\nu_t} \frac{\partial V_s^*(\xi, t)}{\partial t} \quad (83)$$

which when integrated twice gives

$$G_1(\xi, t) = \frac{1}{\nu_t} \int_0^\xi \int_0^{\xi_1} \frac{\partial V_s^*(\xi_2, t)}{\partial t} d\xi_2 d\xi_1 - \frac{1}{h\nu_t} \int_0^h \int_0^\xi \int_0^{\xi_1} \frac{\partial V_s^*(\xi_2, t)}{\partial t} d\xi_2 d\xi_1 d\xi \quad (84)$$

The constant of integration [the last term on the RHS of (84)] follows from the fact that (81) implies that the integral of G_1 over depth is zero. [(81) also implies that $\partial G_1/\partial \xi$ is zero at the bed.] Thus,

$$\begin{aligned} & \sum_{n=1}^{\infty} \cos\left(\frac{n\pi\xi}{h}\right) \int_0^t \frac{dA_\alpha^{(n)}}{dT} \exp[-\lambda_n(t-T)] dT = \\ & \frac{1}{\nu_t} \left[\int_0^\xi \int_0^{\xi_1} \frac{\partial V_s^*(\xi_2, t)}{\partial t} d\xi_2 d\xi_1 \right. \\ & \quad \left. - \int_0^h \int_0^\xi \int_0^{\xi_1} \frac{\partial V_s^*(\xi_2, t)}{\partial t} d\xi_2 d\xi_1 d\xi \right] \\ & - \sum_{n=1}^{\infty} \cos\left(\frac{n\pi\xi}{h}\right) \frac{h^2}{n^2\pi^2\nu_t} \int_0^t \frac{d^2 A_\alpha^{(n)}}{dT^2} \exp[-\lambda_n(t-T)] dT \end{aligned} \quad (85)$$

The process of integrating by parts can be continued indefinitely to lead to the following result

$$\begin{aligned} V_{1\alpha}^{(0,1)} &= V_s^*(\xi, t) \\ &+ \frac{1}{\nu_t} \left[\int_0^\xi \int_0^{\xi_1} \frac{\partial V_s^*(\xi_2, t)}{\partial t} d\xi_2 d\xi_1 - \frac{1}{h} \int_0^h \int_0^\xi \int_0^{\xi_1} \frac{\partial V_s^*(\xi_2, t)}{\partial t} d\xi_2 d\xi_1 d\xi \right] \\ &+ \frac{1}{\nu_t^2} \left[\int_0^\xi \int_0^{\xi_1} \int_0^{\xi_2} \int_0^{\xi_3} \frac{\partial^2 V_s^*(\xi_4, t)}{\partial t^2} d\xi_4 d\xi_3 d\xi_2 d\xi_1 \right. \\ &\quad \left. - \frac{1}{h} \int_0^h \int_0^\xi \int_0^{\xi_1} \int_0^{\xi_2} \int_0^{\xi_3} \frac{\partial^2 V_s^*(\xi_4, t)}{\partial t^2} d\xi_4 d\xi_3 d\xi_2 d\xi_1 d\xi \right] \\ &+ \dots \end{aligned} \quad (86)$$

The structure of the result above suggests a more general solution. For cases in which $h_b^2/\nu_0 T < 1$ (where ν_0 is a typical value of the eddy viscosity), the $\partial/\partial t$ term in

$$\frac{\partial V_{1\alpha}^{(0,1)}}{\partial t} - \frac{\partial}{\partial \xi} \left(\nu_t \frac{\partial V_{1\alpha}^{(0,1)}}{\partial \xi} \right) = R_\alpha^{(0)}(\xi, t) \quad (87)$$

is smaller than the second term on the LHS. Solving (87) using a straightforward perturbation expansion leads to (37). The condition $h_b^2/\nu_0 T < 1$ is satisfied for short-wave-averaged motions (*e.g.*, infragravity waves) in the nearshore.

Finally note that the solution presented here ignores the slow time variation of h in (64). The error caused by this neglect is of order δ and is small.

B Derivation of the integrals

Here we describe the calculations that lead to (44). First, we have (using the definition for $Q_{w\alpha}$)

$$\int_{-h_0}^{\bar{\zeta}} V_{1\alpha}^{(1)} V_{1\beta}^{(0)} dz + V_{1\alpha}^{(1)}(\bar{\zeta}) Q_{w\beta} = \int_{-h_0}^{\bar{\zeta}} [V_{1\alpha}^{(1)}(z) - V_{1\alpha}^{(1)}(\bar{\zeta})] V_{1\beta}^{(0)} dz \quad (88)$$

From (42) we have

$$V_{1\alpha}^{(1)}(z) - V_{1\alpha}^{(1)}(\bar{\zeta}) = - \int_z^{\bar{\zeta}} \frac{1}{\nu_t} \int_{-h_0}^{z'} R_\alpha dz'' dz' \quad (89)$$

which implies that

$$\int_{-h_0}^{\bar{\zeta}} [V_{1\alpha}^{(1)}(z) - V_{1\alpha}^{(1)}(\bar{\zeta})] V_{1\beta}^{(0)} dz = - \int_{-h_0}^{\bar{\zeta}} V_{1\beta}^{(0)} \int_z^{\bar{\zeta}} \frac{1}{\nu_t} \int_{-h_0}^{z'} R_\alpha dz'' dz' dz \quad (90)$$

$$= - \int_{-h_0}^{\bar{\zeta}} \left(\int_{-h_0}^z R_\alpha dz'' \right) \left(\int_{-h_0}^z V_{1\beta}^{(0)} dz' \right) dz \quad (91)$$

Therefore,

$$\begin{aligned} \int_{-h_0}^{\bar{\zeta}} V_{1\alpha} V_{1\beta} dz + V_{1\beta}(\bar{\zeta}) Q_{w\alpha} + V_{1\alpha}(\bar{\zeta}) Q_{w\beta} &= \int_{-h_0}^{\bar{\zeta}} V_{1\alpha}^{(0)} V_{1\beta}^{(0)} dz + V_{1\beta}^{(0)}(\bar{\zeta}) Q_{w\alpha} + V_{1\alpha}^{(0)}(\bar{\zeta}) Q_{w\beta} \\ &\quad - \int_{-h_0}^{\bar{\zeta}} \left(\int_{-h_0}^z R_\alpha dz'' \right) \left(\int_{-h_0}^z V_{1\beta}^{(0)} dz' \right) dz \\ &\quad - \int_{-h_0}^{\bar{\zeta}} \left(\int_{-h_0}^z R_\beta dz'' \right) \left(\int_{-h_0}^z V_{1\alpha}^{(0)} dz' \right) dz \end{aligned} \quad (92)$$

Substituting

$$\begin{aligned} R_\alpha &= \left(\tilde{V}_\delta \frac{\partial V_{1\alpha}^{(0)}}{\partial x_\delta} + V_{1\delta}^{(0)} \frac{\partial \tilde{V}_\alpha}{\partial x_\delta} \right) + W \frac{\partial V_{1\alpha}^{(0)}}{\partial z} + V_{1\delta}^{(0)} \frac{\partial V_{1\alpha}^{(0)}}{\partial x_\delta} \\ &\quad - \frac{1}{h} \frac{\partial}{\partial x_\delta} \left[\int_{-h_0}^{\bar{\zeta}} V_{1\alpha}^{(0)} V_{1\delta}^{(0)} dz + V_{1\delta}^{(0)}(\bar{\zeta}) Q_{w\alpha} + V_{1\alpha}^{(0)}(\bar{\zeta}) Q_{w\delta} \right] \end{aligned} \quad (93)$$

where the vertical velocity W is given by

$$W = - \left[\left(\tilde{V}_\delta + V_{1\delta}^{(0)}(-h_0) \right) \frac{\partial h_0}{\partial x_\delta} + (h_0 + z) \frac{\partial \tilde{V}_\delta}{\partial x_\delta} + \int_{-h_0}^z \frac{\partial V_{1\delta}^{(0)}}{\partial x_\delta} dz \right] \quad (94)$$

into (92) leads to

$$\begin{aligned} \int_{-h_0}^{\bar{\zeta}} V_{1\alpha} V_{1\beta} dz + V_{1\beta}(\bar{\zeta}) Q_{w\alpha} + V_{1\alpha}(\bar{\zeta}) Q_{w\beta} &= M_{\alpha\beta} + A_{\alpha\beta\delta} \tilde{V}_\delta \\ &\quad - h \left(D_{\delta\beta} \frac{\partial \tilde{V}_\alpha}{\partial x_\delta} + D_{\delta\alpha} \frac{\partial \tilde{V}_\beta}{\partial x_\delta} + B_{\alpha\beta} \frac{\partial \tilde{V}_\delta}{\partial x_\delta} \right) \end{aligned} \quad (95)$$

where

$$\begin{aligned} A_{\alpha\beta\delta} &= - \left\{ \int_{-h_0}^{\bar{\zeta}} \frac{1}{\nu_t} \left[\int_{-h_0}^z \frac{\partial V_{1\alpha}^{(0)}}{\partial x_\delta} - \frac{\partial h_0}{\partial x_\delta} \frac{\partial V_{1\alpha}^{(0)}}{\partial z'} dz' \right] \left(\int_{-h_0}^z V_{1\beta}^{(0)} dz'' \right) dz + \right. \\ &\quad \left. \int_{-h_0}^{\bar{\zeta}} \frac{1}{\nu_t} \left[\int_{-h_0}^z \frac{\partial V_{1\beta}^{(0)}}{\partial x_\delta} - \frac{\partial h_0}{\partial x_\delta} \frac{\partial V_{1\beta}^{(0)}}{\partial z'} dz' \right] \left(\int_{-h_0}^z V_{1\alpha}^{(0)} dz'' \right) dz \right\} \end{aligned} \quad (96)$$

$$\begin{aligned} B_{\alpha\beta} &= - \frac{1}{h} \left\{ \int_{-h_0}^{\bar{\zeta}} \frac{1}{\nu_t} \left[\int_{-h_0}^z (h_0 + z') \frac{\partial V_{1\alpha}^{(0)}}{\partial z'} dz' \right] \left(\int_{-h_0}^z V_{1\beta}^{(0)} dz'' \right) dz + \right. \\ &\quad \left. \int_{-h_0}^{\bar{\zeta}} \frac{1}{\nu_t} \left[\int_{-h_0}^z (h_0 + z') \frac{\partial V_{1\beta}^{(0)}}{\partial z'} dz' \right] \left(\int_{-h_0}^z V_{1\alpha}^{(0)} dz'' \right) dz \right\} \end{aligned} \quad (97)$$

$$D_{\alpha\beta} = \frac{1}{h} \int_{-h_0}^{\bar{\zeta}} \frac{1}{\nu_t} \left(\int_{-h_0}^z V_{1\alpha}^{(0)} dz' \right) \left(\int_{-h_0}^z V_{1\beta}^{(0)} dz'' \right) dz \quad (98)$$

$$\begin{aligned} M_{\alpha\beta} &= \int_{-h_0}^{\bar{\zeta}} V_{1\alpha}^{(0)} V_{1\beta}^{(0)} dz + V_{1\beta}^{(0)}(\bar{\zeta}) Q_{w\alpha} + V_{1\alpha}^{(0)}(\bar{\zeta}) Q_{w\beta} \\ &\quad - \left\{ \int_{-h_0}^{\bar{\zeta}} \frac{1}{\nu_t} \left(\int_{-h_0}^z V_{1\delta}^{(0)} \frac{\partial V_{1\beta}^{(0)}}{\partial x_\delta} dz' \right) \left(\int_{-h_0}^z V_{1\alpha}^{(0)} dz' \right) \right. \\ &\quad \left. + \int_{-h_0}^{\bar{\zeta}} \frac{1}{\nu_t} \left(\int_{-h_0}^z V_{1\delta}^{(0)} \frac{\partial V_{1\alpha}^{(0)}}{\partial x_\delta} dz' \right) \left(\int_{-h_0}^z V_{1\beta}^{(0)} dz' \right) \right\} \\ &\quad \frac{\partial h_0}{\partial x_\delta} \left\{ \int_{-h_0}^{\bar{\zeta}} \frac{1}{\nu_t} \left(\int_{-h_0}^z V_{1\delta}^{(0)} \frac{\partial V_{1\beta}^{(0)}}{\partial z'} dz' \right) \left(\int_{-h_0}^z V_{1\alpha}^{(0)} dz' \right) \right. \\ &\quad \left. + \int_{-h_0}^{\bar{\zeta}} \frac{1}{\nu_t} \left(\int_{-h_0}^z V_{1\delta}^{(0)} \frac{\partial V_{1\alpha}^{(0)}}{\partial z'} dz' \right) \left(\int_{-h_0}^z V_{1\beta}^{(0)} dz' \right) \right\} \\ &\quad + \left\{ I_\alpha \int_{-h_0}^{\bar{\zeta}} \frac{(h_0 + z)}{\nu_t} \left(\int_{-h_0}^z V_{1\beta}^{(0)} dz' \right) \right. \\ &\quad \left. + I_\beta \int_{-h_0}^{\bar{\zeta}} \frac{(h_0 + z)}{\nu_t} \left(\int_{-h_0}^z V_{1\alpha}^{(0)} dz' \right) \right\} \end{aligned} \quad (99)$$

In (99) above I_α is defined by

$$I_\alpha = \frac{1}{h} \frac{\partial}{\partial x_\beta} \left[\int_{-h_0}^{\bar{\zeta}} V_{1\alpha}^{(0)} V_{1\beta}^{(0)} dz + V_{1\beta}^{(0)}(\bar{\zeta}) Q_{w\alpha} + V_{1\alpha}^{(0)}(\bar{\zeta}) Q_{w\beta} \right] \quad (100)$$

References

- Blumberg, A.F. & Mellor, G.L. 1987 A description of a three-dimensional coastal ocean circulation model. *Three Dimensional Coastal Ocean Models* (ed. N.S. Heaps), 1-16, AGU.
- Carslaw, H.S. & Jaeger, J.C. 1959 *Conduction of heat in solids*. Oxford University Press.
- Fischer, H.B., List, E.J., Koh, R.C.Y., Imberger, J. & Brooks N.H. 1978 *Mixing in Inland and Coastal Waters*. Academic Press.
- Mei, C.C. 1989 *The applied dynamics of ocean surface waves*. World Scientific.
- Phillips, O.M. 1977 *The dynamics of the upper ocean*. Cambridge University Press.
- Putrevu, U. & Svendsen, I.A. 1992 A mixing mechanism in the nearshore region. *Proc. 23rd International Conference on Coastal Engineering*, 2758-2771.
- Smith, R. 1997 Multi-mode models of flow and of solute dispersion in shallow water. Part 3. Horizontal dispersion tensor for the velocity, *J. Fluid Mech.*, **352**, 331-340.
- Svendsen, I.A. 1984 Mass flux and undertow in a surf-zone. *Coastal Engineering*, **8**, 347-365.
- Svendsen, I.A. & Lorenz, R.S. 1989 Velocities in combined undertow and longshore currents. *Coastal Engineering*, **13**, 55-79.
- Svendsen, I.A. & Putrevu, U. 1994 Nearshore mixing and dispersion. *Proc. Roy. Soc. Lond.*, **A 445**, 561-576.
- Svendsen, I.A. & Putrevu, U. 1996 Surf zone hydrodynamics. *Advances in Coastal and Ocean Engineering*, **2**, 1-78.
- Taylor, G.I. 1953 Dispersion of soluble matter in solvent flowing slowly through a tube. *Proc. Roy. Soc. Lond.*, **A 219**, 186-203.
- Taylor, G.I. 1954 The dispersion of matter in a turbulent flow through a pipe. *Proc. Roy. Soc. Lond.*, **A 219**, 446-468.

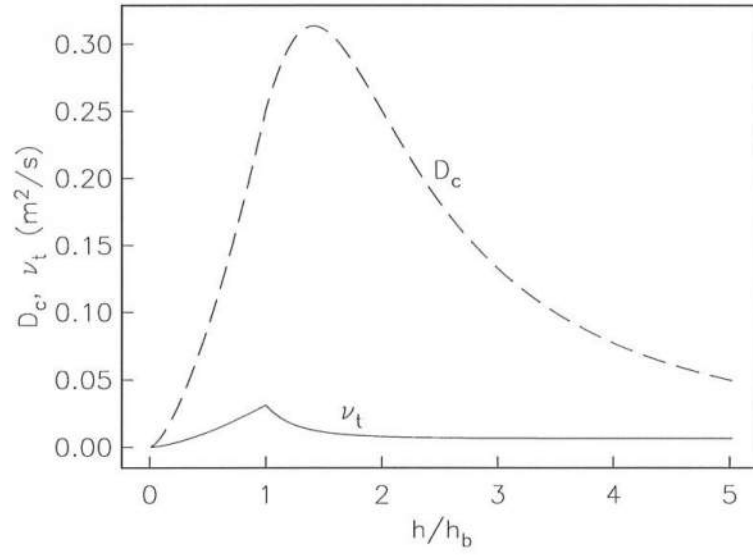


Figure 1: Variations of the turbulent mixing (solid line) and dispersive mixing (dashed line) coefficients as a function of the nondimensional cross-shore distance for the case of steady longshore currents on an alongshore-uniform coast (from Putrevu & Svendsen 1992).

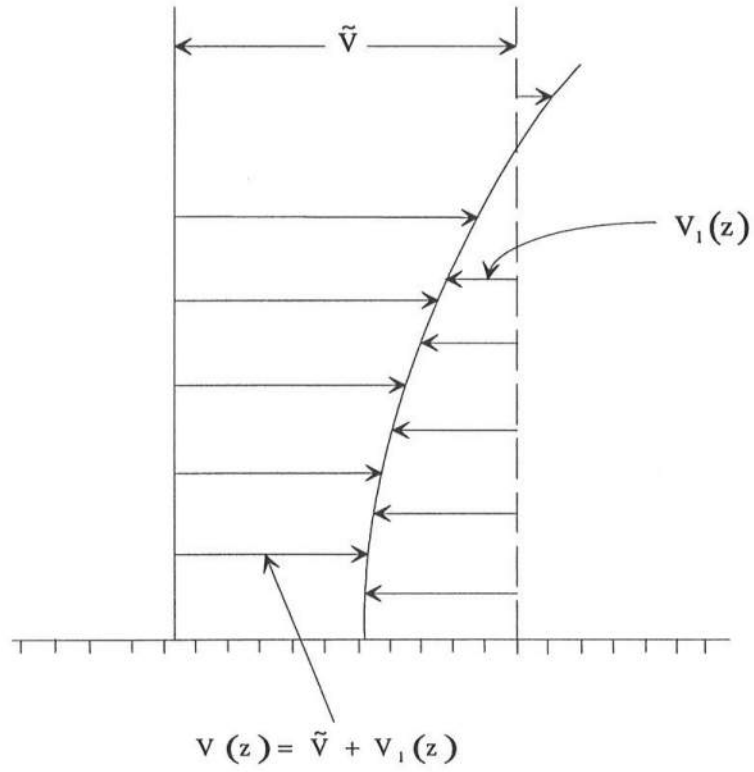


Figure 2: Sketch showing the definitions of \tilde{V} , $V_{1\alpha}(z)$, and $V(z)$. Compare this figure with Figure 3.3 of Phillips (1977).